Homework No. 03 (2019 Fall)

PHYS 301: Theoretical Methods in Physics

Due date: Wednesday, 2019 Sep 4, 10:00 AM, in class

1. Keywords: Analytic function, Cauchy-Riemann conditions, Cauchy's integral formula, Residue theorem, Laurant series.

2. (**30 points.**) Analytic functions are significantly constrained, in that they have to satisfy the Cauchy-Riemann conditions. These conditions are necessary (but not sufficient) for a function of a complex variable to be analytic (differentiable). Check if the following functions satisfy the Cauchy-Riemann conditions.

$$f(z) = z^3, \tag{1a}$$

$$f(z) = |z|^2, \tag{1b}$$

$$f(z) = e^{iz}, \tag{1c}$$

$$f(z) = \ln z. \tag{1d}$$

3. (**20 points.**) Check if the function

$$f(z) = \frac{1}{z} \tag{2}$$

satisfies the Cauchy-Riemann conditions. In particular, investigate when $z = 0$ and $z \neq 0$. Plot

$$f(x) = e^{-\frac{1}{x}}, \tag{3}$$

given x is real. Let z be complex. Check if the function

$$f(z) = e^{-\frac{1}{z}} \tag{4}$$

satisfies the Cauchy-Riemann conditions. In particular, investigate when $z = 0$ and $z \neq 0$.

4. (**10 points.**) Evalauate the following contour integrals. In the following the contour c is a unit circle going counterclockwise with center at the complex number a.

$$I(a) = \frac{1}{2\pi i} \oint_c dz \frac{(z^5 + 1)}{(z - a)}, \tag{5a}$$

$$I(a) = \frac{1}{2\pi i} \oint_c dz \frac{e^{iz}}{(z - a)}. \tag{5b}$$

5. (**20 points.**) Consider the integral

$$I(a) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(1 + a\cos\theta)}, \tag{6}$$

where a is complex. Substitute $z = e^{i\theta}$, such that

$$2\cos\theta = z + \frac{1}{z}, \tag{7}$$

and express the integral as a contour integral,

$$I(a) = \frac{1}{2\pi i} \frac{2}{a} \oint_c \frac{dz}{\left(z^2 + \frac{2}{a}z + 1\right)}, \tag{8}$$

where the contour c is along the unit circle going counterclockwise. Show that

$$z^2 + \frac{2}{a}z + 1 = (z - r_+)(z - r_-), \tag{9}$$

where

$$r_\pm = -\frac{1}{a} \pm \sqrt{\frac{1}{a^2} - 1}. \tag{10}$$

Using residue theorem evaluate $I(a)$ for $|\mathrm{Re}(a)| < 1$ and $\mathrm{Im}(a) = 0$.