# Homework No. 08 (2019 Fall)

## PHYS 301: Theoretical Methods in Physics

Due date: Friday, 2019 Oct 25, 10:00 AM, in class

0A. Keywords: Discrete Fourier transformation, Continuous Fourier transformation, Fourier series, Inverse Fourier tranform, Function space, Special functions, Orthogonality relations, Completeness relation, $\delta$-function, Differential equation for special functions.

0B. Problems 4, 5, and 7, are to be submitted for assessment. Rest are for practice.

1. (**Example.**) A vector $\mathbf{A}$ in three dimensions can be expressed in the form

$$\mathbf{A} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3. \tag{1}$$

Here $\hat{\mathbf{e}}_i$ are called the basis vectors and $a_i$ are components of the vector along the basis vectors.

(a) Orthogonality relation: Let us assume that the basis vectors are orthogonal to each other. This is stated compactly as

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}, \qquad i, j = 1, 2, 3, \tag{2}$$

where $\delta_{ij}$ is the Kronecker delta symbol.

(b) Vector components: Taking the dot product with $\hat{\mathbf{e}}_1$ in each term in Eq. (1) we obtain

$$\mathbf{A} \cdot \hat{\mathbf{e}}_1 = a_1(\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1) + a_2(\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1) + a_3(\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_1). \tag{3}$$

Using the orthogonality relations between the basis vectors we immediately have

$$\mathbf{A} \cdot \hat{\mathbf{e}}_1 = a_1. \tag{4}$$

Similar relations can be derived for other components, and they can be together expressed in the form

$$\mathbf{A} \cdot \hat{\mathbf{e}}_i = a_i, \qquad i = 1, 2, 3. \tag{5}$$

(c) Completeness relation: Substituting the expressions for the vector components back in Eq. (1) we have

$$\mathbf{A} = (\mathbf{A} \cdot \hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1 + (\mathbf{A} \cdot \hat{\mathbf{e}}_2)\hat{\mathbf{e}}_2 + (\mathbf{A} \cdot \hat{\mathbf{e}}_3)\hat{\mathbf{e}}_3 \tag{6a}$$

$$= \mathbf{A} \cdot \Big[\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_3\Big], \tag{6b}$$

where the second equality is obtained by recognizing the common factor. Thus, the vector multiplied with the quantity inside square brackets returns back the vector.

Since the multiplication involves a scalar dot product, the quantity in square brackets can not be a vector because then it will return a scalar. We identify it to be the unit dyadic. Thus,

$$\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_3 = \mathbf{1}, \tag{7}$$

which is the completeness relation for the basis vectors.

2. (**Example.**) The Fourier space is spanned by the Fourier eigenfunctions

$$e^{im\phi}, \qquad m = 0, \pm 1, \pm 2, \ldots, \qquad 0 \le \phi < 2\pi. \tag{8}$$

An arbitrary function $f(\phi)$ has the Fourier series representation

$$f(\phi) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} a_m e^{im\phi}, \tag{9}$$

where $e^{im\phi}$ are the Fourier eigenfunctions and $a_m$ are the respective Fourier components.

(a) Orthogonality relation: The Fourier eigenfunctions satisfy the orthogonality relation

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi\, e^{-in\phi} e^{im\phi} = \delta_{mn}. \tag{10}$$

(b) Fourier components: Using the orthogonality relations we can find the Fourier components to be

$$a_m = \int_0^{2\pi} d\phi\, e^{-im\phi} f(\phi). \tag{11}$$

(c) Completeness relation: The Fourier eigenfunctions satisfy the completeness relation

$$\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im\phi} e^{-im\phi'} = \delta(\phi - \phi'). \tag{12}$$

(d) Differential equation: The Fourier eigenfunctions satisfy the differential equation

$$-\left[\frac{d^2}{d\phi^2} - m^2\right] e^{im\phi} = 0. \tag{13}$$

3. (**Example.**) The (continuous) Fourier space is spanned by the Fourier eigenfunctions

$$e^{ikx}, \qquad -\infty < k < \infty, \qquad -\infty < x < \infty. \tag{14}$$

An arbitrary function $f(x)$ has the Fourier series representation

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k), \tag{15}$$

where $e^{ikx}$ are the Fourier eigenfunctions and $\tilde{f}(k)$ are the respective Fourier components.

(a) Orthogonality relation: The Fourier eigenfunctions satisfy the orthogonality relation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{-ik'x} e^{ikx} = \delta(k-k'). \tag{16}$$

(b) Fourier components: Using the orthogonality relations we can find the Fourier components to be

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x). \tag{17}$$

(c) Completeness relation: The Fourier eigenfunctions satisfy the completeness relation

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} e^{-ikx'} = \delta(x-x'). \tag{18}$$

(d) Differential equation: The Fourier eigenfunctions satisfy the differential equation

$$-\left[\frac{d^2}{dx^2} - k^2\right] e^{ikx} = 0. \tag{19}$$

4. (**20 points.**) Fourier series (or transformation) is defined as $(0 \leq \phi < 2\pi)$

$$f(\phi) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{im\phi} a_m, \tag{20}$$

where the coefficients $a_m$ are determined using

$$a_m = \int_0^{2\pi} d\phi\, e^{-im\phi} f(\phi). \tag{21}$$

Determine the particular function $f(\phi)$ which leads to

$$a_m = 1 \tag{22}$$

for all $m$. That is, all the Fourier coefficients are contributing equally in the series.

5. (**20 points.**) Fourier series (or transformation) is defined as $(0 \leq \phi < 2\pi)$

$$f(\phi) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{im\phi} a_m, \tag{23}$$

where the coefficients $a_m$ are determined using

$$a_m = \int_0^{2\pi} d\phi\, e^{-im\phi} f(\phi). \tag{24}$$

Determine all the Fourier components $a_m$ for the following functions: $\cos\phi$, $\sin\phi$, $\cos^2\phi$, $\sin^2\phi$, $\cos^3\phi$, $\sin^3\phi$.

6. (**20 points.**) To determine the Fourier components of $\tan\phi$ start from

$$\tan\phi = \frac{1}{i}\frac{e^{i\phi} - e^{-i\phi}}{e^{i\phi} + e^{-i\phi}} \tag{25}$$

and show that

$$\tan\phi = \frac{1}{i} + \sum_{m=1}^{\infty} e^{-2im\phi}\frac{2(-1)^m}{i}. \tag{26}$$

Thus, read out all the Fourier components. Similarly, find the Fourier components of $\cot\phi$.

7. (**20 points.**) Fourier series (or transformation) is defined as $(-\infty < x < \infty)$

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} a(k), \tag{27}$$

where the coefficients $a(k)$ are determined using

$$a(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x). \tag{28}$$

(a) Show that

$$\frac{d^n f(x)}{dx^n} = \int_{-\infty}^{\infty} \frac{dk}{2\pi}(ik)^n e^{ikx} a(k). \tag{29}$$

(b) Show that the differential equation

$$-\left(\frac{d^2}{dx^2} - \omega^2\right) f(x) = \delta(x) \tag{30}$$

in the Fourier space is the algebraic equation

$$(k^2 + \omega^2) a(k) = 1. \tag{31}$$

Thus, the solution to the differential equation is the Fourier transform of

$$a(k) = \frac{1}{\omega^2 + k^2}. \tag{32}$$

8. (**20 points.**) Consider the inhomogeneous linear differential equation

$$\left(a\frac{d^2}{dx^2} + b\frac{d}{dx} + c\right) f(x) = \delta(x). \tag{33}$$

Use the Fourier transformation and the associated inverse Fourier transformation

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k), \tag{34a}$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x), \tag{34b}$$

to show that the corresponding equation satisfied by $\tilde{f}(k)$ is algebraic. Find $\tilde{f}(k)$.