Homework No. 07 (Fall 2019)

PHYS 520A: Electromagnetic Theory I

Due date: Monday, 2019 Nov 11, 4.00pm

1. (**10 points.**) Verify the identity

$$\phi \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} \psi) - \psi \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} \phi) = \boldsymbol{\nabla} \cdot [\lambda(\phi \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \phi)], \tag{1}$$

which is a slight generalization of what is known as Green's second identity. Here ϕ, ψ, and λ, are position dependent functions.

2. (**10 points.**) Show that the potential for a point charge, in three spatial dimensions,

$$\phi(\mathbf{r}) = \frac{q_a}{4\pi\varepsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_a|}, \tag{2}$$

satisfies the differential equation

$$-\varepsilon_0 \nabla^2 \phi(\mathbf{r}) = q_a \delta^{(3)}(\mathbf{r} - \mathbf{r}_a). \tag{3}$$

Solve the corresponding differential equation in one spatial dimension,

$$-\varepsilon_0 \frac{d^2}{dx^2} \phi(x) = q_a \delta(x - x_a). \tag{4}$$

Thus, apparently, the electric potential between two charges, in 1 space+1 time dimensions goes linearly with the distance r between the charges, while it goes inversely in 3 space+1 time dimensions. This leads to the conclusion that two like charges will attract and two unlike charges will repel in 1 space+1 time dimensions. This is called the Schwinger model. Read about the the Schwinger model and write a few sentences on it.

Hints:

(a) Using the definition of δ-function observe that

$$-\varepsilon_0 \frac{d^2}{dx^2} \phi(x) = 0, \quad \text{for} \quad x \neq x_a. \tag{5}$$

(b) Solve the homogeneous differential equation in Eq. (5) in terms of two integral constants in each of two regions,

$$\phi(x) = \begin{cases} a_1 x + b_1, & x < x_a, \\ a_2 x + b_2, & x > x_a. \end{cases} \tag{6}$$

(c) Integrate Eq. (4) from $x = x_a - \delta$ to $x = x_a + \delta$, for infinitesimal $\delta > 0$, to derive the boundary condition on

$$\frac{d}{dx}\phi(x). \tag{7}$$

(d) Argue that, for consistency, we also require the boundary condition

$$\phi(x_a - \delta) = \phi(x_a + \delta). \tag{8}$$

(e) Use the boundary conditions to determine two of the four integral constants in Eq. (6). In particular find $a_2 - a_1$ and $b_2 - b_1$. The solutions can be expressed in the form

$$\phi(x) = -\frac{q}{2\varepsilon_0}|x - x_a| + ax + b, \tag{9}$$

where $2a = a_1 + a_2$ and $2b = b_1 + b_2$.