Homework No. 01 (2020 Spring)

PHYS 301: THEORETICAL METHODS IN PHYSICS

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Due date: None

0. Keywords: Hyperbolic functions, trignometric functions in the complex plane, inverse hyperbolic functions.

0. Reference: Chapter X in Advanced Trigonometry, by Durell and Robson, which is a popular high school textbook.

1. Hyperbolic cosine function and sine function are defined using the exponential function. We have

$$\cosh x = \frac{e^x + e^{-x}}{2}, \tag{1a}$$
$$\sinh x = \frac{e^x - e^{-x}}{2}. \tag{1b}$$

Here, and in the following, assume x and y to be real. Recall that the corresponding trignometric functions are defined in terms of the exponential function as

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \tag{2a}$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}. \tag{2b}$$

Hyperbolic functions extend the domain of the corresponding trignometric functions to the complex plane.

(a) Show that

$$\cosh x = \cos(ix), \tag{3a}$$
$$\sinh x = -i\sin(ix). \tag{3b}$$

(b) Plot $\cosh x$ and $\sinh x$ as functions of x.

(c) Using Eqs. (1) derive the identity

$$\cosh^2 x - \sinh^2 x = 1. \tag{4}$$

Derive the identities for the sum of arguments of hyperbolic functions,

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y, \tag{5a}$$
$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y. \tag{5b}$$

Derive the derivative operations on hyperbolic functions,

$$\frac{d}{dx}\cosh x = \sinh x, \tag{6a}$$

$$\frac{d}{dx}\sinh x = \cosh x, \tag{6b}$$

and the integral operations,

$$\int dx \, \cosh x = \sinh x, \tag{7a}$$

$$\int dx \, \sinh x = \cosh x. \tag{7b}$$

(d) To find the inverse hyperbolic function of sine let us define

$$y = \sinh^{-1} x. \tag{8}$$

Then, we have

$$x = \sinh y = \frac{e^y - e^{-y}}{2}, \tag{9}$$

which can be rewritten as a quadratic equation in e^y,

$$(e^y)^2 - 2x(e^y) - 1 = 0, \tag{10}$$

with solutions

$$e^y = x \pm \sqrt{x^2 + 1}. \tag{11}$$

Presuming y to be real argue that only one of the roots is consistent with $e^y > 0$. Taking logarithm we have

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}). \tag{12}$$

Similarly, derive the expression

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}). \tag{13}$$