Homework No. 04 (2020 Spring)

PHYS 301: THEORETICAL METHODS IN PHYSICS

Department of Physics, Southern Illinois University–Carbondale

Due date: Monday, 2020 Feb 10, 9:00 AM, in class

0. Problems 3, 5, and 8, are to be submitted for assessment. Rest are for practice.

0. Keywords: Index notation; Dyadic notation; Kronecker delta-symbol; Levi-Civita symbol; Vector calculus.

1. (**10 points.**) Verify the following relations:

$$\delta_{ij} = \delta_{ji}, \tag{1a}$$
$$\delta_{ii} = 3, \tag{1b}$$
$$\delta_{ik}\delta_{kj} = \delta_{ij}, \tag{1c}$$
$$\delta_{im}B_m = B_i, \tag{1d}$$
$$\varepsilon_{ijk} = -\varepsilon_{ikj} = \varepsilon_{kij}, \tag{1e}$$
$$\varepsilon_{iik} = 0, \tag{1f}$$
$$\delta_{ij}\varepsilon_{ijk} = 0. \tag{1g}$$

2. (**10 points.**) In three dimensions the Levi-Civita symbol is given in terms of the determinant of the Kronecker δ-functions,

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \tag{2a}$$
$$= = \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}). \tag{2b}$$

Using the above identity show that

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}, \tag{3a}$$
$$\varepsilon_{ijk}\varepsilon_{ijn} = 2\delta_{kn}, \tag{3b}$$
$$\varepsilon_{ijk}\varepsilon_{ijk} = 6. \tag{3c}$$

3. (**10 points.**) Using the property of Kronecker δ-function and Levi-Civita symbol evaluate the following using index notation.

$$\delta_{ij}\delta_{ji} = \tag{4a}$$
$$\delta_{ij}\varepsilon_{ijk} = \tag{4b}$$
$$\varepsilon_{ijm}\delta_{mn}\varepsilon_{nij} = \tag{4c}$$

4. (**10 points.**) Using index notation and the antisymmetric property of the Levi-Civita symbol show that

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{B}\cdot\mathbf{C}\times\mathbf{A} = -\mathbf{A}\cdot\mathbf{C}\times\mathbf{B}. \tag{5}$$

5. (**10 points.**) Derive the following vector identities (using index notation)

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}), \tag{6}$$

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}), \tag{7}$$

6. (**20 points.**) Verify the following identities:

$$\boldsymbol{\nabla} r = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}, \tag{8a}$$

$$\boldsymbol{\nabla}\,\mathbf{r} = \mathbf{1}. \tag{8b}$$

Further, show that

$$\boldsymbol{\nabla}\cdot\mathbf{r} = 3, \tag{9a}$$

$$\boldsymbol{\nabla}\times\mathbf{r} = 0. \tag{9b}$$

Here r is the magnitude of the position vector $\mathbf{r}$, and $\hat{\mathbf{r}}$ is the unit vector pointing in the direction of $\mathbf{r}$.

7. (**10 points.**) (Based on Problem 1.13, Griffiths 4th edition.)
Show that

$$\boldsymbol{\nabla} r^2 = 2\mathbf{r}. \tag{10}$$

Then evaluate $\boldsymbol{\nabla} r^3$. Show that

$$\boldsymbol{\nabla}\frac{1}{r} = -\frac{\hat{\mathbf{r}}}{r^2}. \tag{11}$$

Then evaluate $\boldsymbol{\nabla}(1/r^2)$.

8. (**10 points.**) Evaluate the left hand side of the equation

$$\boldsymbol{\nabla}(\mathbf{r}\cdot\mathbf{p}) = a\,\mathbf{p} + b\,\mathbf{r}, \tag{12}$$

where $\mathbf{p}$ is a constant vector. Thus, find a and b.

9. (**20 points.**) Evaluate the left hand side of the equation

$$\boldsymbol{\nabla}\cdot(r^2\mathbf{r}) = a\,r^n. \tag{13}$$

Thus, find a and n.