Homework No. 07 (2020 Spring)

PHYS 301: THEORETICAL METHODS IN PHYSICS

Department of Physics, Southern Illinois University–Carbondale

Due date: Monday, 2020 Mar 2, 9:00 AM, in class

0. Problems 1 and 3 are to be submitted for assessment. Rest are for practice.

0. Keywords: Eigenvalues and eigenvectors of a matrix; Matrix diagonalization; Properties of Pauli matrices; Eigenbasis dependence of matrices.

1. (**30 points.**) A particular representation of Pauli matrices is

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

(In particular, these are Pauli matrices in the eigenbasis of σ_z.) Find the eigenvalues, normalized eigenvectors, and diagonalizing matrix, for each of the three Pauli matrix. Verify that your results satisfy the eigenvalue equation.

2. (**20 points.**) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}. \tag{2}$$

   (a) Find the eigenvalues of the matrix $\mathbf{A}$.

   (b) Find the normalized eigenvectors of matrix $\mathbf{A}$.

   (c) Determine the matrix that diagonalizes the matrix $\mathbf{A}$.

   (d) What can you then conclude about the eigenvalues and eigenvectors of $\ln \mathbf{A}$? Find them.

3. (**20 points.**) Consider the rotation matrix

$$\mathbf{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{3}$$

   (a) Find the eigenvalues of the matrix $\mathbf{A}$.

   (b) Find the normalized eigenvectors of matrix $\mathbf{A}$.

   (c) Determine the matrix that diagonalizes the matrix $\mathbf{A}$.

   (d) What can you then conclude about the eigenvalues and eigenvectors of $\mathbf{A}^{107}$? Find them.

4. (**20 points.**) Consider the matrix

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \tag{4}$$

(a) Find all the eigenvalues of the matrix A.

(b) Find the normalized eigenvectors associated with all the eigenvalues of matrix A. (Simplification is achieved by writing the trignometric functions in terms of half angles. $1 - \cos\theta = 2\sin^2\theta/2$, $1 + \cos\theta = 2\cos^2\theta/2$, $\sin\theta = 2\sin\theta/2\cos\theta/2$.)

(c) Determine the matrix that diagonalizes the matrix A.

5. (**20 points.**) Construct the matrix

$$\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}, \tag{5}$$

where

$$\boldsymbol{\sigma} = \sigma_x \hat{\mathbf{i}} + \sigma_y \hat{\mathbf{j}} + \sigma_z \hat{\mathbf{k}}, \tag{6}$$

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}. \tag{7}$$

Use the representation of Pauli matrices is

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8}$$

Find the eigenvalues of the matrix $\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}$.

6. (**20 points.**) A 3×3 matrix A satisfies the equation

$$A^3 = 1. \tag{9}$$

Given that the eigenvalues of A are non-degenerate, find all eigenvalues of A.