Homework No. 09 (2020 Spring)

PHYS 301: THEORETICAL METHODS IN PHYSICS

Department of Physics, Southern Illinois University–Carbondale

Due date: Monday, 2020 Mar 30, 9:00 AM, in class

0. Problems 3 and 4 are to be submitted for assessment. Rest are for practice.

0. Keywords: Differential equation for damped harmonic oscillator.

1. (**Example.**) A damped harmonic oscillator, constituting of a body of mass m and a spring of spring constant k, is described by

$$ma = -kx - bv, \tag{1}$$

where x is position, $v = dx/dt$ is velocity, $a = dv/dt$ is acceleration, and b is the damping coefficient. Thus, we have the differential equation

$$\left[\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2\right] x(t) = 0 \tag{2}$$

with initial conditions

$$x(0) = x_0, \tag{3a}$$
$$\dot{x}(0) = v_0, \tag{3b}$$

where

$$\omega_0^2 = \frac{k}{m}, \qquad 2\gamma = \frac{b}{m}. \tag{4}$$

(a) $\gamma = 0$: In the absence of damping show that the solution is

$$x(t) = x_0 \cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t. \tag{5}$$

(b) $\gamma < \omega_0$: Underdamped harmonic oscillator.

$$x(t) = e^{-\gamma t}\left[x_0 \cos\sqrt{\omega_0^2 - \gamma^2}\,t + \frac{(v_0 + \gamma x_0)}{\sqrt{\omega_0^2 - \gamma^2}}\sin\sqrt{\omega_0^2 - \gamma^2}\,t\right]. \tag{6}$$

(c) $\gamma = \omega_0$: Critically damped harmonic oscillator.

$$x(t) = e^{-\omega_0 t}\left[x_0 + (v_0 + \omega_0 x_0)t\right]. \tag{7}$$

(d) $\gamma > \omega_0$: Overdamped harmonic oscillator.

$$x(t) = e^{-\gamma t}\left[x_0 \cosh\sqrt{\gamma^2-\omega_0^2}\,t + \frac{(v_0+\gamma x_0)}{\sqrt{\gamma^2-\omega_0^2}}\sinh\sqrt{\gamma^2-\omega_0^2}\,t\right]. \tag{8}$$

(e) Set $\omega_0 = 1$, which is equivalent to the substitution $\omega_0 t = \tau$, and sets the scale for the time t. That is, time is measured in units of $T = 2\pi/\omega_0$. The system is then completely characterized by the parameter γ/ω_0 and the initial conditions x_0 and v_0. Plot the solutions for the initial conditions $x_0 = 0$ and $v_0 = 1$.

2. (**20 points.**) A critically damped harmonic oscillator is described by the differential equation

$$\left[\frac{d^2}{dt^2} + 2\omega_0\frac{d}{dt} + \omega_0^2\right]x(t) = 0, \tag{9}$$

where ω_0 is a characteristic frequency. Find the solution $x(t)$ for initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$. Plot $x(t)$ as a function of t in the following graph where $x_0\,e^{-\omega_0 t}$ is already plotted for reference. For what t is the solution $x(t)$ a maximum?

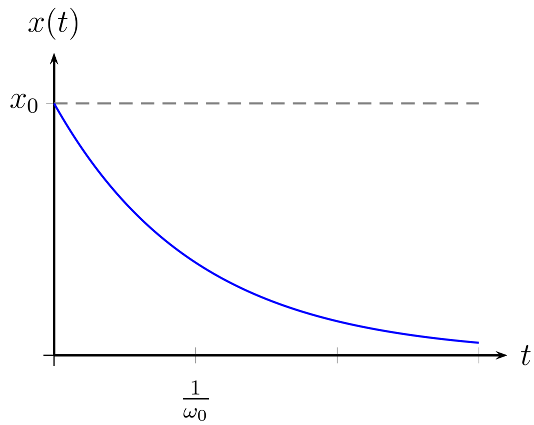

Figure 1: Critically damped harmonic oscillator.

3. (**20 points.**) A body experiencing only damping is described by the differential equation

$$\left[\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt}\right]x(t) = 0, \tag{10}$$

where γ is a measure of the damping. Find the solution $x(t)$ for initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$ to be

$$x(t) = x_0 + \frac{v_0}{2\gamma}\left[1 - e^{-2\gamma t}\right]. \tag{11}$$

Obtain the above expression starting from the solution for the overdamped harmonic oscillator ($\gamma > \omega_0$)

$$x(t) = e^{-\gamma t}\left[x_0 \cosh\sqrt{\gamma^2-\omega_0^2}\,t + \frac{(v_0+\gamma x_0)}{\sqrt{\gamma^2-\omega_0^2}}\sinh\sqrt{\gamma^2-\omega_0^2}\,t\right] \tag{12}$$

by setting $\omega_0 = 0$. Interpret the solution for $v_0 = 0$, why isn't there no motion?

4. (**20 points.**) Starting from the solution for the underdamped harmonic oscillator ($\gamma < \omega_0$),

$$x(t) = e^{-\gamma t}\left[x_0 \cos\sqrt{\omega_0^2 - \gamma^2}t + \frac{(v_0 + \gamma x_0)}{\sqrt{\omega_0^2 - \gamma^2}} \sin\sqrt{\omega_0^2 - \gamma^2 t}\right], \tag{13}$$

obtain the solution for the overdamped harmonic oscillator ($\gamma > \omega_0$),

$$x(t) = e^{-\gamma t}\left[x_0 \cosh\sqrt{\gamma^2 - \omega_0^2}t + \frac{(v_0 + \gamma x_0)}{\sqrt{\gamma^2 - \omega_0^2}} \sinh\sqrt{\gamma^2 - \omega_0^2 t}\right]. \tag{14}$$