Midterm Exam No. 02 (2020 Spring)

PHYS 420: Electricity and Magnetism II

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1. (**20 points.**) The magnetic field due to a point magnetic dipole $\mathbf{m}$ at a distance $\mathbf{r}$ away from the point magnetic dipole is given by the expression

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\big[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\big]}{r^3}, \qquad r \neq 0. \tag{1}$$

Let there be two point magnetic dipoles of equal strength. The first is positioned at the origin, $\mathbf{r}_1 = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{k}}$, and is pointing in the $-\hat{\mathbf{x}}$ direction, $\mathbf{m}_1 = -m\,\hat{\mathbf{x}}$. The second is positioned on the z axis a distance $2a$ from the origin, $\mathbf{r}_2 = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{i}} + 2a\,\hat{\mathbf{k}}$, and is pointing in the $\hat{\mathbf{z}}$ direction, $\mathbf{m}_2 = m\,\hat{\mathbf{z}}$.

   (a) Find the magnitude and direction of the magnetic field at the position $\mathbf{r} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{i}} + a\,\hat{\mathbf{k}}$,

   (b) Qualitatively plot the magnetic field lines in regions very far ($a \ll r$) from the dipoles.

2. (**20 points.**) Given

$$\mathbf{N} = \frac{\hat{\boldsymbol{\phi}}}{\rho^2} \tag{2}$$

and the gradient operator

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \tag{3}$$

in cylindrical coordinates. Evaluate

$$\boldsymbol{\nabla} \times \mathbf{N}. \tag{4}$$

3. (**20 points.**) The complete elliptic integrals have the power series expansions

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n} = \frac{\pi}{2}\left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \ldots\right], \tag{5a}$$

$$E(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 \frac{k^{2n}}{(1-2n)} = \frac{\pi}{2}\left[1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \ldots\right]. \tag{5b}$$

The leading order contribution in the power series expansions are from $K(0)$ and $E(0)$. Evaluate the leading order contribution of

$$\left[K(k) - \frac{(2-k^2)}{2(1-k^2)}E(k)\right]. \tag{6}$$

Hint: Truncate all series expansions to order k^0 and collect the terms. If it is zero, repeat for order k^2. Repeat for subsequent higher orders until you obtain a non-zero contribution.

4. (**20 points.**) The flux of electromagnetic momentum density (the stress tensor) is

$$\mathbf{T} = \mathbf{1}U - (\mathbf{ED} + \mathbf{HB}), \tag{7}$$

where

$$U = \frac{1}{2}\mathbf{E}\cdot\mathbf{D} + \frac{1}{2}\mathbf{H}\cdot\mathbf{B} \tag{8}$$

is the electromagnetic energy density. Here $\mathbf{D} = \varepsilon_0\mathbf{E}$ and $\mathbf{B} = \mu_0\mathbf{H}$. Evaluate

$$\hat{\mathbf{z}}\cdot\mathbf{T}\cdot\hat{\mathbf{z}} \tag{9}$$

in a region where

$$\mathbf{E} = E_0\hat{\mathbf{x}} \quad \text{and} \quad \mathbf{B} = 0. \tag{10}$$