Homework No. 01 (2020 Spring)

PHYS 520B: ELECTROMAGNETIC THEORY

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Due date: Tuesday, 2020 Jan 21, 4.30pm

1. (**20 points.**) The solution to the Maxwell equations for the case of magnetostatics in terms of the vector potential $\mathbf{A}$ is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{1}$$

   (a) Verify that the above solution satisfies the Coulomb gauge condition. That is, it satisfies

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0. \tag{2}$$

   (b) Further, verify that the magnetic field is the curl of the vector potential and can be expressed in the form

$$\mathbf{B}(\mathbf{r}) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{3}$$

2. (**20 points.**) The solution to the Maxwell equations for the case of magnetostatics was found to be

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{4}$$

   Verify that the above solution satisfies magnetostatics equations, that is, it satisfies

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{5}$$

   and

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{6}$$

3. (**50 points.**) (Based on Problem 5.8, Griffiths 4th edition.)
   The magnetic field at position $\mathbf{r} = (x, y, z)$ due to a finite wire segment of length $2L$ carrying a steady current I, with the caveat that it is unrealistic (why?), placed on the z-axis with its end points at $(0, 0, L)$ and $(0, 0, -L)$, is

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{4\pi} \frac{1}{\sqrt{x^2+y^2}} \left[\frac{z+L}{\sqrt{x^2+y^2+(z+L)^2}} - \frac{z-L}{\sqrt{x^2+y^2+(z-L)^2}} \right], \tag{7}$$

   where $\hat{\boldsymbol{\phi}} = (-\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}) = (-y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}})/\sqrt{x^2+y^2}$.

(a) Show that by taking the limit $L \to \infty$ we obtain the magnetic field near a long straight wire carrying a steady current I,

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{2\pi\rho}, \tag{8}$$

where $\rho = \sqrt{x^2 + y^2}$ is the perpendicular distance from the wire.

(b) Show that the magnetic field on a line bisecting the wire segment is given by

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{2\pi\rho}\frac{L}{\sqrt{\rho^2 + L^2}}. \tag{9}$$

(c) Find the magnetic field at the center of a square loop, which carries a steady current I. Let $2L$ be the length of a side, ρ be the distance from center to side, and $R = \sqrt{\rho^2 + L^2}$ be the distance from center to a corner. (Caution: Notation differs from Griffiths.) You should obtain

$$B = \frac{\mu_0 I}{2R}\frac{4}{\pi}\tan\frac{\pi}{4}. \tag{10}$$

(d) Show that the magnetic field at the center of a regular n-sided polygon, carrying a steady current I is

$$B = \frac{\mu_0 I}{2R}\frac{n}{\pi}\tan\frac{\pi}{n}, \tag{11}$$

where R is the distance from center to a corner of the polygon.

(e) Show that the magnetic field at the center of a circular loop of radius R,

$$B = \frac{\mu_0 I}{2R}, \tag{12}$$

is obtained in the limit $n \to \infty$.

4. (**40 points.**) (Refer Schwinger et al. problem 26.1 and the article in Ref. [1].) A simple model of a metal describes the electrons in it using Newton's law,

$$m\frac{d^2\mathbf{x}}{dt^2} + m\gamma\frac{d\mathbf{x}}{dt} + m\omega_0^2\mathbf{x} = e\mathbf{E}. \tag{13}$$

Here the first term involves the acceleration of electron, ω_0-term binds the electron to the atoms, while γ-term damps the motion.

Conductor: Conductivity in typical metals is dominated by the damping term, thus

$$m\gamma\mathbf{v} = e\mathbf{E}. \tag{14}$$

The current density $\mathbf{j}$ for (constant) density n_f of conduction electrons is

$$\mathbf{j} = n_f e\mathbf{v}. \tag{15}$$

Using Eqs. (14) and (15) in conjunction we have Ohm's law

$$\mathbf{j} = \frac{n_f e^2}{m\gamma}\mathbf{E} = \sigma\mathbf{E}, \tag{16}$$

where σ is the static conductivity.

Superconductor: In 1935 Fritz London and Heinz London proposed that the current density $\mathbf{j}_s$ in a superconductor is described by the acceleration term in Eq. (13). That is,

$$m\frac{d\mathbf{v}}{dt} = e\mathbf{E}, \tag{17}$$

which together with Eq. (15) leads to London "acceleration equation"

$$\frac{d\mathbf{j}_s}{dt} = \frac{n_f e^2}{m}\mathbf{E}. \tag{18}$$

As a consequence steady currents are possible solutions when $\mathbf{E} = 0$. The insight of the London brothers led them to further propose, in addition, that the current density in a superconductor satisfies

$$\boldsymbol{\nabla} \times \left(\mathbf{j}_s + \frac{n_f e^2}{m}\mathbf{A}\right) = 0. \tag{19}$$

Thus, up to a freedom in the choice of gauge χ, we have the London equation

$$\mu_0 \mathbf{j}_s = -\frac{1}{\lambda_L^2}\Big(\mathbf{A} + \boldsymbol{\nabla}\chi\Big), \tag{20}$$

where λ_L defined using

$$\frac{n_f e^2}{m} = \frac{1}{\lambda_L^2}\frac{1}{\mu_0} \tag{21}$$

is the London penetration depth which is a measure of the distance magnetic field penetrates into the surface of a superconductor. The London equation replaces Ohm's law for a superconductor. Note that the London equation is consistent with the "acceleration equation" using the gauge freedom

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi, \tag{22}$$

$$\phi' = \phi - \frac{\partial\chi}{\partial t}. \tag{23}$$

(a) Using London's equation show that a superconductor is characterized by the equations

$$\mu_0\frac{\partial\mathbf{j}_s}{\partial t} = \frac{1}{\lambda_L^2}\mathbf{E}, \tag{24}$$

$$\mu_0\boldsymbol{\nabla}\times\mathbf{j}_s = -\frac{1}{\lambda_L^2}\mathbf{B}. \tag{25}$$

(b) Show that the magnetic field satisfies the equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B} = \frac{1}{\lambda_L^2}\mathbf{B}. \tag{26}$$

For the static case, $\partial\mathbf{B}/\partial t = 0$, show that

$$\nabla^2\mathbf{B} = \frac{1}{\lambda_L^2}\mathbf{B}, \tag{27}$$

which implies the Meissner effect, that a uniform magnetic field cannot exist inside a superconductor. In this static limit, and presuming planar geometry, it implies

$$\mathbf{B} = \mathbf{B}_0\, e^{-\frac{x}{\lambda_L}}, \tag{28}$$

where the interpretation of λ_L as a penetration depth is apparent. Using Eq. (21) calculate the penetration depth for $n_f \sim 6 \times 10^{28}\,/\mathrm{m}^3$ (electron number density for gold) and show that it is of the order of tens of nanometers.

References

[1] F. London and H. London. The electromagnetic equations of the supraconductor. *Proc. R. Soc. London A: Mathematical, Physical and Engineering Sciences*, 149(866):71–88, 1935.