Homework No. 02 (2020 Spring)

PHYS 520B: ELECTROMAGNETIC THEORY

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Due date: Friday, 2020 Jan 30, 4.30pm

1. (**20 points.**) The magnetic field for a straight wire of infinite extent carrying a steady current I is given by
$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{2\pi\rho}. \tag{1}$$
Verify that
$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \tag{2}$$
everywhere. In particular, investigate if the magnetic field is divergenceless on the wire, where $\rho = 0$. Next, evaluate
$$\boldsymbol{\nabla}\times\mathbf{B} \tag{3}$$
everywhere. Thus, check if the magnetic field due to a straight current carrying wire satisfies the two Maxwell equations relevant for magnetostatics.

2. (**20 points.**) The vector potential for a point magnetic moment $\mathbf{m}$ is given by
$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3}. \tag{4}$$
Verify that the magnetic field due to the point dipole obtained by evaluating the curl
$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} \tag{5}$$
can be expressed in the form
$$\mathbf{B}(\mathbf{r}) = \mathbf{m}\,\mu_0\,\delta^{(3)}(\mathbf{r}) + \frac{\mu_0}{4\pi}(\mathbf{m}\cdot\boldsymbol{\nabla})\left(\boldsymbol{\nabla}\frac{1}{r}\right). \tag{6}$$
Verify that the magnetic field satisfies the Maxwell equation
$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0. \tag{7}$$
Show that
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\Big[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\Big] + \mathbf{m}\,\mu_0\,\delta^{(3)}(\mathbf{r}). \tag{8}$$

3. (**40 points.**) A charged spherical shell carries a charge q. It rotates with angular velocity $\boldsymbol{\omega}$ about a diameter.

(a) Show that the current density generated by this motion is given by

$$\mathbf{J}(\mathbf{r}) = \frac{q}{4\pi a^2}\,\boldsymbol{\omega} \times \mathbf{r}\,\delta(r-a). \tag{9}$$

Hint: Use $\mathbf{J}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}$ and $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ for circular motion.

(b) Using

$$\mathbf{m} = \frac{1}{2}\int d^3r\,\mathbf{r} \times \mathbf{J}(\mathbf{r}). \tag{10}$$

determine the magnetic dipole moment of the rotating sphere to be

$$\mathbf{m} = \frac{qa^2}{3}\boldsymbol{\omega}. \tag{11}$$

(c) Evaluate the vector potential inside and outside the sphere to be

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0}{4\pi}\dfrac{\mathbf{m} \times \mathbf{r}}{a^3}, & r < a, \\[2ex] \dfrac{\mu_0}{4\pi}\dfrac{\mathbf{m} \times \mathbf{r}}{r^3}, & a < r. \end{cases} \tag{12}$$

Hint: Out of the three vectors $\boldsymbol{\omega}$, the observation point $\mathbf{r}$, and the integration variable $\mathbf{r}'$, choose $\mathbf{r}$ to be along the z axis while working in spherical polar coordinates. This leads to considerable simplification in the expression for $|\mathbf{r} - \mathbf{r}'|$ appearing in the denominator. Otherwise, without choosing $\mathbf{r}$ to be along $\hat{\mathbf{z}}$, use the ideas of Legendre polynomials and spherical harmonics.

(d) Derive the corresponding expression for the magnetic field, using $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, to be

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0}{4\pi}\,2\mathbf{m}, & r < a, \\[2ex] \dfrac{\mu_0}{4\pi}\dfrac{1}{r^3}\Big[3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\Big], & a < r. \end{cases} \tag{13}$$