Homework No. 03 (2020 Spring)

PHYS 520B: ELECTROMAGNETIC THEORY

Department of Physics, Southern Illinois University–Carbondale

Due date: Thursday, 2020 Feb 6, 4.30pm

0. Problems 1, 4, and 6, are to be submitted for assessment. Rest are for practice and some will be covered in lectures.

1 Complete elliptic integrals

Complete elliptic integrals of the first and second kind can be defined using the integral representations,

$$K(k) = \int_0^{\frac{\pi}{2}} d\psi \frac{1}{\sqrt{1-k^2\sin^2\psi}}, \tag{1a}$$

$$E(k) = \int_0^{\frac{\pi}{2}} d\psi \sqrt{1-k^2\sin^2\psi}, \tag{1b}$$

respectively.

1. (**20 points.**) Verify that

$$K(0) = \frac{\pi}{2}, \tag{2a}$$

$$E(0) = \frac{\pi}{2}. \tag{2b}$$

Then, verify that

$$E(1) = 1. \tag{3}$$

Note that

$$K(1) = \int_0^{\frac{\pi}{2}} \frac{d\psi}{\cos\psi} \tag{4}$$

is devergent. To see the nature of divergence we introduce a cutoff parameter $\delta > 0$ and write

$$K(1) = \int_0^{\frac{\pi}{2}-\delta} \frac{d\psi}{\cos\psi}. \tag{5}$$

Evaluate the integral, (using the identity $d(\sec\psi+\tan\psi)/d\psi = \sec\psi(\sec\psi+\tan\psi)$,) and show that

$$K(1) \sim \ln 2 - \ln\delta - \frac{\delta^2}{12} + \mathcal{O}(\delta)^4 \tag{6}$$

has logarithmic divergence. Using Mathematica (or another graphing tool) plot $K(k)$ and $E(k)$ as functions of k for $0 \leq k < 1$.

2. (**20 points.**) The complete elliptic integrals have the power series expansions

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n} = \frac{\pi}{2}\left[1+\frac{1}{4}k^2+\frac{9}{64}k^4+\ldots\right], \tag{7a}$$

$$E(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n} = \frac{\pi}{2}\left[1-\frac{1}{4}k^2-\frac{3}{64}k^4-\ldots\right]. \tag{7b}$$

The leading order contribution in the power series expansions are from $K(0)$ and $E(0)$. Evaluate the next-to-leading order contributions in the above series expansions by expanding the radical in Eqs.(1) as a series. Use

$$\frac{1}{\sqrt{1-x}} = 1+\frac{1}{2}x+\ldots, \tag{8a}$$

$$\sqrt{1-x} = 1-\frac{1}{2}x+\ldots. \tag{8b}$$

3. (**20 points.**) Show that the perimeter of an ellipse, characterized by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1, \tag{9}$$

with eccentricity

$$e = \sqrt{1-\frac{b^2}{a^2}}, \tag{10}$$

is given by

$$C = 4aE(e), \tag{11}$$

where $E(k)$ is the complete elliptic integralof the second kind,

$$E(k) = \int_0^{\frac{\pi}{2}} d\psi\sqrt{1-k^2\sin^2\psi}. \tag{12}$$

A circle is an ellipse with zero eccentricity. Deduce the circumference of a circle using the formula.

4. (**50 points.**) (Refer Landau and Lifshitz, Problem 1 in Chapter 3.)
A simple pendulum, consisting of a particle of mass m suspended by a string of length l in a uniform gravitational field g, is described by the Hamiltonian

$$H = \frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi. \tag{13}$$

(a) For initial conditions $\phi(0) = \phi_0$ and $\dot{\phi}(0) = 0$ show that

$$\frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi = -mgl\cos\phi_0. \tag{14}$$

Thus, derive

$$\frac{dt}{T_0} = \frac{1}{2\pi} \frac{d\phi}{\sqrt{2(\cos\phi - \cos\phi_0)}} \tag{15}$$

where $T_0 = 2\pi\sqrt{l/g}$.

(b) Determine the period of oscillations of the simple pendulum as a function of the amplitude of oscillations ϕ_0 to be

$$T = T_0 \frac{2}{\pi} K\left(\sin\frac{\phi_0}{2}\right), \tag{16}$$

where

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} \tag{17}$$

is the complete elliptic integral of the first kind. Hint: Substitute

$$\sin\theta = \frac{\sin\frac{\phi}{2}}{\sin\frac{\phi_0}{2}}. \tag{18}$$

(c) Using the power series expansion

$$K(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n} \tag{19}$$

show that for small oscillations ($\phi_0/2 \ll 1$)

$$T = T_0 \left[1 + \frac{\phi_0^2}{16} + \ldots\right]. \tag{20}$$

(d) Estimate the percentage error made in the approximation $T \sim T_0$ for $\phi_0 \sim 60°$.

(e) Plot the time period T of Eq. (16) as a function of ϕ_0. What can you conclude about the time period for $\phi_0 = \pi$?

2 Exact result in terms of elliptic integrals

5. (**30 points.**) The current density for a circular loop of radius a carrying a steady current I is given by

$$\mathbf{j}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\, I\delta(\rho - a)\delta(z), \tag{21}$$

where the the loop is chosen to be in the x-y plane with the origin as its center.

(a) Verify that

$$\int_S d\mathbf{a} \cdot \mathbf{j} = I, \tag{22}$$

where surface S is a half-plane of constant ϕ.

(b) Show that magnetic vector potential is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} a \int_0^{2\pi} d\phi' \frac{\hat{\boldsymbol{\phi}}'}{\sqrt{z^2 + \rho^2 + a^2 - 2\rho a \cos(\phi - \phi')}}. \tag{23}$$

(c) Substitute $\phi' - \phi \to \phi'$ and show that

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{4\pi} a \int_0^{2\pi} d\phi' \frac{\cos\phi'}{\sqrt{z^2 + \rho^2 + a^2 - 2\rho a \cos\phi'}}. \tag{24}$$

(d) The ϕ' integral can not be completed in terms of elementary functions. Show that in terms of the complete elliptic integrals of the first and second kind,

$$K(k) = \int_0^{\frac{\pi}{2}} d\psi \frac{1}{\sqrt{1 - k^2 \sin^2\psi}}, \tag{25a}$$

$$E(k) = \int_0^{\frac{\pi}{2}} d\psi \sqrt{1 - k^2 \sin^2\psi}, \tag{25b}$$

respectivly, the magnetic vector potential is

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{4\pi} \frac{4a}{\sqrt{z^2 + (\rho + a)^2}} \left[\frac{2}{k^2} \Big\{ K(k) - E(k) \Big\} - K(k) \right], \tag{26}$$

where

$$k^2 = \frac{4a\rho}{z^2 + (\rho + a)^2}. \tag{27}$$

Hint: Show that the contributions to the ϕ' integral in Eq. (24) gets equal contributions from 0 to π and π to 2π. In particular, use the form with $(z^2+\rho^2+a^2+2\rho a\cos\phi')$ in the denominator. Then, use the half-angle formula to obtain the integral in terms of the complete elliptic integrals.

6. (**30 points.**) We have earlier found the magnetic vector potential to be zero everywhere along the symmetry axis of the circular loop. With our exact expression let us calculate an approximate expression for the magnetic vector potential very close to the axis. Using the power series expansions for the complete elliptic integrals show that

$$\frac{2}{k^2} \Big\{ K(k) - E(k) \Big\} - K(k) = \frac{\pi}{16} k^2 + \ldots. \tag{28}$$

Drop the next-to-leading order terms, valid when $k \ll 1$, and show that

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\, A(\rho, z) = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{4\pi} \frac{a^2 \pi \rho}{[z^2 + (\rho + a)^2]^{\frac{3}{2}}}. \tag{29}$$

Check that $\mathbf{A} = 0$ on the axis. Show that the magnetic field close to the axis ($k \ll 1$) is given by

$$\mathbf{B}(\mathbf{r}) = -\hat{\boldsymbol{\rho}} \frac{\partial A}{\partial z} + \hat{\mathbf{z}} \left(\frac{\partial}{\partial \rho} + \frac{1}{\rho} \right) A. \tag{30}$$

7. (**20 points.**) The expression for the magnetic vector potential $\mathbf{A}$ and the magnetic field $\mathbf{B}$ for a circular loop of radius a carrying a current I is given in terms of the complete elliptic integrals. An approximate expression for the magnetic vector potential close to the axis is

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\, A(\rho, z) = \hat{\boldsymbol{\phi}}\, \frac{\mu_0 I}{4\pi} \frac{a^2 \pi \rho}{[z^2 + (\rho + a)^2]^{\frac{3}{2}}}. \tag{31}$$

Check that $\mathbf{A} = 0$ on the axis. The magnetic field close to the axis, then, is calculated using

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{32}$$

Show that the magnetic field close to the axis ($k \ll 1$) is given by

$$\mathbf{B}(\mathbf{r}) = -\hat{\boldsymbol{\rho}} \frac{\partial A}{\partial z} + \hat{\mathbf{z}} \left(\frac{\partial A}{\partial \rho} + C \right). \tag{33}$$

Find C.

8. (**30 points.**) The current density for a circular loop of radius a carrying a steady current I is given by

$$\mathbf{j}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\, I \delta(\rho - a) \delta(z), \tag{34}$$

where the the loop is chosen to be in the x-y plane with the origin as its center.

(a) Using Bio-Savart law and completing the integrals involving δ-functions show that magnetic field has the form

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} d\phi' \frac{\left[a^2 \hat{\mathbf{z}} + az \hat{\boldsymbol{\rho}}' - a\rho (\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\phi}}') \right]}{[z^2 + \rho^2 + a^2 - 2\rho a \cos(\phi - \phi')]^{\frac{3}{2}}}. \tag{35}$$

(b) Substitute $\phi' - \phi \to \phi'$ and show that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} d\phi' \frac{\left[(a^2 - a\rho \cos\phi') \hat{\mathbf{z}} + az \hat{\boldsymbol{\rho}} \cos\phi' + az \hat{\boldsymbol{\phi}} \sin\phi' \right]}{[z^2 + \rho^2 + a^2 - 2\rho a \cos\phi']^{\frac{3}{2}}}. \tag{36}$$

(c) The ϕ' integral can not be completed in terms of elementary functions. Show that in terms of the complete elliptic integrals of the first and second kind,

$$K(k) = \int_0^{\frac{\pi}{2}} d\psi \frac{1}{\sqrt{1 - k^2 \sin^2 \psi}}, \tag{37a}$$

$$E(k) = \int_0^{\frac{\pi}{2}} d\psi \sqrt{1 - k^2 \sin^2 \psi}, \tag{37b}$$

respectivly, the magnetic field is

$$\begin{aligned} \mathbf{B}(\mathbf{r}) = {} & \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \frac{2}{\sqrt{z^2 + (\rho + a)^2}} \left[K(k) - \frac{(z^2 + \rho^2 - a^2)}{z^2 + (\rho - a)^2} E(k) \right] \\ & - \hat{\boldsymbol{\rho}} \frac{\mu_0 I}{4\pi} \frac{2}{\sqrt{z^2 + (\rho + a)^2}} \frac{z}{\rho} \left[K(k) - \frac{(z^2 + \rho^2 + a^2)}{z^2 + (\rho - a)^2} E(k) \right], \end{aligned} \tag{38}$$

where

$$k^2 = \frac{4a\rho}{z^2 + (\rho + a)^2}. \tag{39}$$

Hint: Show that the contributions to the ϕ' integral in Eq. (24) gets equal contributions from 0 to π and π to 2π. In particular, use the form with $(z^2+\rho^2+a^2+2\rho a \cos\phi')$ in the denominator. Then, use the half-angle formula to obtain the integral in terms of the complete elliptic integrals. It is useful to identify

$$\int_0^{\frac{\pi}{2}} d\psi \frac{1}{(1 - k^2 \sin^2\psi)^{\frac{3}{2}}} = \frac{E(k)}{(1 - k^2)}. \tag{40}$$