Homework No. 04 (2020 Spring)

PHYS 520B: ELECTROMAGNETIC THEORY

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2020 Feb 25, 4.30pm

1. (**50 points.**) Hamiltonian for a charge particle of mass m and charge q in a magnetic field $\mathbf{B}$ is given by

$$H(\mathbf{x},\mathbf{p}) = \frac{1}{2m}\left(\mathbf{p} - q\mathbf{A}\right)^2, \tag{1}$$

where

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}. \tag{2}$$

Let

$$\frac{\partial \mathbf{A}}{\partial t} = 0. \tag{3}$$

(a) Show that the Hamilton equations of motion leads to the equations, using ($\mathbf{v} = d\mathbf{x}/dt$)

$$m\mathbf{v} = \mathbf{p} - q\mathbf{A}, \tag{4a}$$

$$\frac{d\mathbf{p}}{dt} = q(\boldsymbol{\nabla}\mathbf{A})\cdot\mathbf{v}. \tag{4b}$$

Show that the above equations in conjunction imply the familiar equation

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v}\times\mathbf{B}. \tag{5}$$

(b) For two functions

$$A = A(\mathbf{x},\mathbf{p},t), \tag{6a}$$

$$B = B(\mathbf{x},\mathbf{p},t), \tag{6b}$$

the Poisson braket with respect to the canonical variables $\mathbf{x}$ and $\mathbf{p}$ is defined as

$$\big[A,B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} \equiv \frac{\partial A}{\partial \mathbf{x}}\cdot\frac{\partial B}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}}\cdot\frac{\partial B}{\partial \mathbf{x}}. \tag{7}$$

Evaluate the Poisson braket

$$\big[\mathbf{x},\mathbf{x}\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0. \tag{8}$$

(c) Evaluate the Poisson braket

$$\big[\mathbf{x}^i,\mathbf{v}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \frac{1}{m}\mathbf{1}^{ij}. \tag{9}$$

(d) Evaluate the Poisson braket

$$\big[\mathbf{x}^i, \mathbf{p}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \mathbf{1}^{ij}. \tag{10}$$

(e) Evaluate the Poisson braket

$$\big[m\mathbf{v}^i, m\mathbf{v}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = q(\boldsymbol{\nabla}^i\mathbf{A}^j - \boldsymbol{\nabla}^i\mathbf{A}^j). \tag{11}$$

Verify that

$$(\boldsymbol{\nabla}^i\mathbf{A}^j - \boldsymbol{\nabla}^i\mathbf{A}^j) = \varepsilon^{ijk}\mathbf{B}^k = -\mathbf{1}\times\mathbf{B}. \tag{12}$$

Poisson bracket in classical mechanics has direct correspondence to commutation relation in quantum mechanics through the factor $i\hbar$, which conforms with experiments and balances the dimensions. Then, we can write

$$\big[m\mathbf{v}, m\mathbf{v}\big] = i\hbar q\mathbf{B} \tag{13}$$

or

$$m\mathbf{v}\times m\mathbf{v} = i\hbar q\mathbf{B}, \tag{14}$$

using the fact that the commutator and the vector product satisfies the same Lie algebra as that of Poisson bracket.