Homework No. 11 (2020 Fall)

PHYS 320: ELECTRICITY AND MAGNETISM I

Due date: Tuesday, 2020 Nov 4, 2:00 PM, on D2L

0. Keywords: Green's function, method of images for planar geometry.

0. Problems 1 and 2 are to be submitted for assessment. Rest are for practice.

1. (**40 points.**) The expression for the electric potential due to a point charge placed in front of a perfectly conducting semi-infinite slab described by

$$\varepsilon(z) = \begin{cases} \infty, & z < 0, \\ \varepsilon_0, & 0 < z, \end{cases} \tag{1}$$

is given in terms of the reduced Green's function that satisfies the differential equation $(0 < \{z, z'\})$

$$-\left[\frac{\partial^2}{\partial z^2} - k^2\right] \varepsilon_0 g(z, z') = \delta(z - z') \tag{2}$$

with boundary conditions requiring the reduced Green's function to vanish at $z = 0$.

(a) Construct the reduced Green's function in the form

$$g(z, z') = \begin{cases} Ae^{kz} + Be^{-kz}, & 0 < z < z', \\ Ce^{kz} + De^{-kz}, & 0 < z' < z, \end{cases} \tag{3}$$

and solve for the four coefficients, A, B, C, D, using the conditions

$$g(0, z') = 0, \tag{4a}$$

$$g(a, z') = 0, \tag{4b}$$

$$g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} = 0, \tag{4c}$$

$$\varepsilon_0 \partial_z g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} = -1. \tag{4d}$$

(b) Express the solution the form

$$g(z, z') = \frac{1}{\varepsilon_0}\frac{1}{2k} e^{-k|z-z'|} - \frac{1}{\varepsilon_0}\frac{1}{2k} e^{-k|z|} e^{-k|z'|}. \tag{5}$$

(c) Deduce the method of images from the above solution.

2. (**40 points.**) The expression for the electric potential due to a point charge placed in between two perfectly conducting semi-infinite slabs described by

$$\varepsilon(z) = \begin{cases} \infty, & z < 0, \\ \varepsilon_0, & 0 < z < a, \\ \infty, & a < z, \end{cases} \tag{6}$$

is given in terms of the reduced Green's function that satisfies the differential equation $(0 < \{z, z'\} < a)$

$$\left[-\frac{\partial^2}{\partial z^2} + k^2\right] \varepsilon_0 g(z, z') = \delta(z - z') \tag{7}$$

with boundary conditions requiring the reduced Green's function to vanish at $z = 0$ and $z = a$.

(a) Construct the reduced Green's function in the form

$$g(z, z') = \begin{cases} A \sinh kz + B \cosh kz, & 0 < z < z' < a, \\ C \sinh kz + D \cosh kz, & 0 < z' < z < a, \end{cases} \tag{8}$$

and solve for the four coefficients, A, B, C, D, using the conditions

$$g(0, z') = 0, \tag{9a}$$

$$g(a, z') = 0, \tag{9b}$$

$$g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} = 0, \tag{9c}$$

$$\varepsilon_0 \partial_z g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} = -1. \tag{9d}$$

Hint: The hyperbolic functions here are defined as

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad \text{and} \qquad \cosh x = \frac{1}{2}(e^x + e^{-x}). \tag{10}$$

(b) Take the limit $ka \to \infty$ in your solution above, (which corresponds to moving the slab at $z = a$ to infinity,) to obtain the reduced Green's function for a single perfectly conducting slab,

$$\lim_{ka\to\infty} g(z, z') = \frac{1}{\varepsilon_0}\frac{1}{2k} e^{-k|z-z'|} - \frac{1}{\varepsilon_0}\frac{1}{2k} e^{-k|z|} e^{-k|z'|}. \tag{11}$$

This should serve as a check for your solution to the reduced Green's function.

3. (**40 points.**) Consider the differential equation

$$\left[-\frac{\partial}{\partial z}\varepsilon(z)\frac{\partial}{\partial z} + \varepsilon(z) k_\perp^2\right] g_\varepsilon(z, z') = \delta(z - z'), \tag{12}$$

for the case

$$\varepsilon(z) = \begin{cases} \varepsilon_2, & z < 0, \\ \varepsilon_1, & 0 < z, \end{cases} \tag{13}$$

satisfying the boundary conditions

$$g_\varepsilon(-\infty, z') = 0, \tag{14a}$$
$$g_\varepsilon(+\infty, z') = 0. \tag{14b}$$

(a) Verify, by integrating Eq. (12) around $z = z'$, that the Green function satisfies the continuity conditions

$$g_\varepsilon(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} = 0, \tag{15a}$$
$$\varepsilon(z)\frac{\partial}{\partial z} g_\varepsilon(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} = -1. \tag{15b}$$

(b) Verify, by integrating Eq. (12) around $z = 0$, that the Green function satisfies the continuity conditions

$$g_\varepsilon(z, z')\Big|_{z=0-\delta}^{z=0+\delta} = 0, \tag{16a}$$
$$\varepsilon(z)\frac{\partial}{\partial z} g_\varepsilon(z, z')\Big|_{z=0-\delta}^{z=0+\delta} = 0. \tag{16b}$$

(c) For $z' < 0$, construct the solution in the form

$$g_\varepsilon(z, z') = \begin{cases} A_1 e^{k_\perp z} + B_1 e^{-k_\perp z}, & z < z' < 0, \\ C_1 e^{k_\perp z} + D_1 e^{-k_\perp z}, & z' < z < 0, \\ E_1 e^{k_\perp z} + F_1 e^{-k_\perp z}, & z' < 0 < z. \end{cases} \tag{17}$$

Determine the constants using the boundary conditions and continuity conditions.

(d) For $0 < z'$, construct the solution in the form

$$g_\varepsilon(z, z') = \begin{cases} A_2 e^{k_\perp z} + B_2 e^{-k_\perp z}, & z < 0 < z', \\ C_2 e^{k_\perp z} + D_2 e^{-k_\perp z}, & 0 < z < z', \\ E_2 e^{k_\perp z} + F_2 e^{-k_\perp z}, & 0 < z' < z. \end{cases} \tag{18}$$

Determine the constants using the boundary conditions and continuity conditions.

(e) Thus, find the solution

$$g_\varepsilon(z, z') = \begin{cases} \dfrac{1}{\varepsilon_2}\dfrac{1}{2k_\perp} e^{-k_\perp |z-z'|} + \dfrac{1}{\varepsilon_2}\dfrac{1}{2k_\perp}\left(\dfrac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1}\right) e^{-k_\perp |z|} e^{-k_\perp |z'|}, & z' < 0, \\ \dfrac{1}{\varepsilon_1}\dfrac{1}{2k_\perp} e^{-k_\perp |z-z'|} + \dfrac{1}{\varepsilon_1}\dfrac{1}{2k_\perp}\left(\dfrac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right) e^{-k_\perp |z|} e^{-k_\perp |z'|}, & 0 < z'. \end{cases} \tag{19}$$