Final Exam (Fall 2020)

PHYS 520A: ELECTROMAGNETIC THEORY I

Department of Physics, Southern Illinois University–Carbondale

Date: Wednesday, 2020 Dec 9, 4:45pm

1. (**20 points.**) Conducting electrons, unlike bound electrons, are not confined to a particular atom. In the Drude model the motion of the conduction electrons are described by Newton's law

$$m\frac{d}{dt}\mathbf{v}(t) = -m\gamma\mathbf{v}(t) + e\mathbf{E}(t), \tag{1}$$

where the effects of collisions are modeled by a frictional force proportional (and opposite) to the velocity. If n_f is the (uniform) density of (free) conduction electrons, then the conduction current density is given by

$$\mathbf{J}(t) = n_f e\mathbf{v}(t). \tag{2}$$

(a) Solve the differential equation in Eq. (1) and express the solution in the form

$$\mathbf{v}(t) = \frac{e}{m}\int_{-\infty}^{t} dt'\, e^{-\gamma(t-t')}\mathbf{E}(t'). \tag{3}$$

Then, using Eq. (2) express this response in the form

$$\mathbf{J}(t) = \int_{-\infty}^{\infty} dt'\, \sigma(t-t')\varepsilon_0\mathbf{E}(t'), \tag{4}$$

where

$$\sigma(t) = \omega_p^2\,\theta(t)\,e^{-\gamma t} \tag{5}$$

and ω_p is the plasma frequency defined using

$$\omega_p^2 = \frac{n_f e^2}{m\varepsilon_0}. \tag{6}$$

(b) Transform the response in Eq. (4) into the frequency space to obtain the statement of Ohm's law

$$\mathbf{J}(\omega) = \sigma(\omega)\varepsilon_0\mathbf{E}(\omega), \tag{7}$$

where the conductivity $\sigma(\omega)$ is determined by the Fourier transformation

$$\sigma(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\sigma(t). \tag{8}$$

Complete the integration Eq. (8), using Eq. (5), to yield the Drude model for conductivity

$$\sigma(\omega) = \frac{\omega_p^2}{\gamma - i\omega}. \tag{9}$$

(c) For a constant electric field

$$\mathbf{E}(t) = \mathbf{E}_0 \tag{10}$$

evaluate the integral in Eq. (4), using Eq. (5), and show that the current density is a constant, given by

$$\mathbf{J}(t) = \frac{\omega_p^2}{\gamma}\varepsilon_0\mathbf{E}_0. \tag{11}$$

Use the Fourier transformation

$$\mathbf{J}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\mathbf{J}(t) \tag{12}$$

to deduce

$$\mathbf{J}(\omega) = \frac{\omega_p^2}{\gamma}2\pi\delta(\omega)\varepsilon_0\mathbf{E}_0. \tag{13}$$

Thus, identify the expression for static conductivity

$$\sigma(\omega) = \frac{\omega_p^2}{\gamma}2\pi\delta(\omega) = \frac{n_f e^2}{m\varepsilon_0}\frac{1}{\gamma}2\pi\delta(\omega). \tag{14}$$

The static conductivity corresponds to response at zero frequency, $\sigma(0)$.

(d) Find the current density for a pulse of infinitely short duration

$$\mathbf{E}(t) = \mathbf{e}_0\delta(t) \tag{15}$$

if $\mathbf{J}(t) = 0$ for $t < 0$. Using Eq. (4) with Eq. (5) show that

$$\mathbf{J}(t) = \omega_p^2\theta(t)e^{-\gamma t}\varepsilon_0\mathbf{e}_0. \tag{16}$$

In particular, determine $\mathbf{J}(t)$ immediately after $t = 0$. Use the Fourier transformation to show that the frequency response is given by

$$\mathbf{J}(\omega) = \frac{\omega_p^2}{\gamma - i\omega}\varepsilon_0\mathbf{e}_0. \tag{17}$$

2. (**20 points.**) The charge density of a low pressure electric arc maintained using a hot filament is a called plasma. Plasma oscillations or Langmuir waves in a dilute plasma are oscillations in an electric arc described by

$$m\mathbf{a} = e\mathbf{E}(t), \tag{18}$$

where we have assumed negligible friction and binding force. Using the current density

$$\mathbf{J}(\mathbf{r}, t) = n_f e\mathbf{v}(t) \tag{19}$$

show that

$$\frac{\partial}{\partial t}\mathbf{J}(\mathbf{r}, t) = \omega_p^2\varepsilon_0\mathbf{E}(\mathbf{r}, t). \tag{20}$$

Taking the divergence in the above equation, and then using the Maxwell equation and the equation of current conservation, deduce the relation for charge density in a dilute plasma to be

$$\frac{\partial^2}{\partial t^2}\rho(\mathbf{r},t) = -\omega_p^2 \rho(\mathbf{r},t) \tag{21}$$

whose solutions describe oscillations with angular frequency ω_p.

3. (**20 points.**) The constitutive relations in a nondispersive media are

$$\mathbf{D} = \varepsilon \mathbf{E}, \tag{22a}$$

$$\mathbf{B} = \mu \mathbf{H}, \tag{22b}$$

where ε and μ are constants. The ratio of speed of light in vacuum c to speed of light in the medium v is the refractive index of the medium

$$n = \frac{c}{v} = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}. \tag{23}$$

The theory of relativity states that velocity of energy flow can not be larger than the speed of light in vacuum. Thus, $n > 1$. Let $\mu = \mu_0$. Consider the dielectric model

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}. \tag{24}$$

This is a complex number, which means a complex velocity of propagation v and a complex index of refraction

$$n = n_r + i n_i = \frac{c}{v} = \sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}}. \tag{25}$$

A complex refractive index signifies that the propagation is accompanied by absorption

$$e^{-i\omega\left(t - \frac{x}{v}\right)} = e^{-i\omega\left(t - n\frac{x}{c}\right)} = e^{-n_i\frac{\omega}{c}x} e^{-i\omega\left(t - n_r\frac{x}{c}\right)}. \tag{26}$$

Thus, c/n_r plays the role of phase velocity and $n_i\omega/c$ is a coefficient of absorption. Plot n_r as a function of ω and verify that it crosses the line $n = 1$ near $\omega = \omega_0$. Thus, apparently, signal in a dispersive medium violates causality. This contradiction was resolved by Sommerfeld and Brillouin in 1914. Translated versions of their papers have been published in a book titled 'Wave Propagation and Group Velocity' by Brillouin in 1960. The book is available at https://archive.org. Very briefly present the resolution here.

4. (**20 points.**) Show that the speed of energy flow of a monochromatic electromagnetic wave in a dispersive medium (for slowly evolving field) when both ε and μ are frequency dependent is given by

$$\frac{v_E}{c} = \left[\frac{d}{d\omega}\left(\omega\sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}\right)\right]^{-1}. \tag{27}$$

Determine the speed of energy flow for the case

$$\mu = \mu_0 \qquad \text{and} \qquad \frac{\varepsilon}{\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2} \tag{28}$$

to be

$$\frac{v_E}{c} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} < 1. \tag{29}$$