Midterm Exam No. 02 (Fall 2020)

PHYS 520A: ELECTROMAGNETIC THEORY I

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1. (**20 points, in class.**) Determine the electric potential inside a spherical shell of radius R that has a total charge Q uniformly distributed on its surface.

2. (**20 points, in class.**) The electromagnetic energy density U and the corresponding energy flux vector $\mathbf{S}$ are given by, ($\mathbf{D} = \varepsilon_0 \mathbf{E}$, $\mathbf{B} = \mu_0 \mathbf{H}$, $\varepsilon_0 \mu_0 c^2 = 1$,)
$$U = \frac{1}{2}(\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}), \qquad \mathbf{S} = \mathbf{E} \times \mathbf{H}. \tag{1}$$
The electromagnetic momentum density $\mathbf{G}$ and the corresponding momentum flux tensor $\mathbf{T}$ are given by
$$\mathbf{G} = \mathbf{D} \times \mathbf{B}, \qquad \mathbf{T} = \mathbf{1}U - (\mathbf{DE} + \mathbf{BH}). \tag{2}$$
Express the trace of the momentum flux tensor in the form
$$\mathrm{Tr}(\mathbf{T}) = aU, \tag{3}$$
and determine a.

3. (**20 points, in class.**) The potential energy of an electric dipole $\mathbf{p}$ in an electric field, that is not necessarily uniform, is
$$U = -\mathbf{p} \cdot \mathbf{E}. \tag{4}$$
Restricting to electrostatics, ($\boldsymbol{\nabla} \cdot \mathbf{D} = \rho$ and $\boldsymbol{\nabla} \times \mathbf{E} = 0$,) show that the force on the electric dipole moment
$$\mathbf{F} = -\boldsymbol{\nabla} U \tag{5}$$
is given in terms of the directional derivative of the electric field in the direction of the electric dipole moment,
$$\mathbf{F} = (\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{E}. \tag{6}$$

4. (**20 points, take home.**) Consider a parallel plate capacitor with plates of infinite extent, with uniform electric field inside the plates,
$$\mathbf{E}(\mathbf{r}) = \begin{cases} \hat{\mathbf{z}} E, & \text{inside}, \\ 0, & \text{outside}. \end{cases} \tag{7}$$

The magnetic field $\mathbf{B} = 0$ everywhere. Starting from the equation for conservation of electromagnetic linear momentum,

$$\frac{\partial \mathbf{G}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{T} + \mathbf{f} = 0, \tag{8}$$

evaluate the force per unit area on the plates of the capacitor.

5. (**20 points, take home.**) We have three charges q_1, q_2, and q_3, at positions $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, respectively. If the configuration has zero electric monopole moment and zero electric dipole moment, then show that the three charges are collinear. Further, show that the electric quadrupole moment of the configuration is

$$\mathbf{q} = q_h \Big[3(\mathbf{r}_1 - \mathbf{r}_2)(\mathbf{r}_1 - \mathbf{r}_2) - (\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{r}_1 - \mathbf{r}_2) \mathbf{1} \Big]. \tag{9}$$

where q_h is the harmonic mean of q_1 and q_2 given by

$$\frac{1}{q_h} = \frac{1}{q_1} + \frac{1}{q_2}. \tag{10}$$

6. (**20 points, take home.**) Let us consider a uniformly charged circular disc of radius a and total charge Q. Let the disc be infinitely thin. Let it be placed on the x-y plane with its center at the origin.

   (a) Show that the charge density for the disc in spherical coordinates can be expressed in the form

   $$\rho(\mathbf{r}') = \frac{Q}{\pi a^2} \frac{\delta\left(\theta' - \frac{\pi}{2}\right)}{r'} \theta(a - r'). \tag{11}$$

   Verify that $\int d^3 r' \rho(\mathbf{r}') = Q$.

   (b) Using symmetry argue that the electric potential has no dependence in the azimuth angle ϕ. Thus,

   $$\phi(\mathbf{r}) = \phi(r, \theta). \tag{12}$$

   Our goal here will be to obtain a solution for the electric potential as an expansion in Legendre polynomials.

   (c) Starting from

   $$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3 r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \tag{13}$$

   find the solution for the electric potential on the z axis (where $\theta = 0$) to be

   $$\phi(r, 0) = \frac{1}{4\pi\varepsilon_0} \frac{2Q}{a^2} \left[\sqrt{a^2 + r^2} - r \right]. \tag{14}$$

   Using the binomial expansion

   $$\sqrt{1 + x^2} = 1 + \sum_{n=1}^{\infty} x^{2n} \frac{(-1)^{n-1}}{2n} \frac{[2(n-1)]!}{2^{2(n-1)}[(n-1)!]^2} \tag{15}$$

express the electric potential on the z axis in the form

$$\phi(r,0) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{a}\left[1-\dfrac{r}{a}+\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{2n}\dfrac{[2(n-1)]!}{2^{2(n-1)}[(n-1)!]^2}\left(\dfrac{r}{a}\right)^{2n}\right], & r<a, \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{r}\displaystyle\sum_{n=0}^{\infty}\dfrac{(-1)^n}{2(n+1)}\dfrac{(2n)!}{2^{2n}(n!)^2}\left(\dfrac{a}{r}\right)^{2n}, & a<r. \end{cases} \tag{16}$$

(d) Let the Legendre expansion of the electric potential be

$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{2Q}{a}\sum_{l=0}^{\infty}A_l(r)P_l(\cos\theta). \tag{17}$$

The electric potential satisfies the Laplacian

$$-\nabla^2\phi = 0 \tag{18}$$

outside the disc. Using the Laplacian in spherical coordinates and the differential equation satisfied by the Legendre polynomials, deduce the differential equation for the coefficients $A_l(r)$ to be

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right]A_l(r) = 0. \tag{19}$$

Show that

$$A_l(r) = \alpha_l\left(\frac{r}{a}\right)^l + \beta_l\left(\frac{a}{r}\right)^{l+1}. \tag{20}$$

Thus, the Legendre expansion for the electric potential is

$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{2Q}{a}\sum_{l=0}^{\infty}\left[\alpha_l\left(\frac{r}{a}\right)^l + \beta_l\left(\frac{a}{r}\right)^{l+1}\right]P_l(\cos\theta). \tag{21}$$

Requiring the boundary condition that the electric potential be zero for $r\to\infty$ and is finite at $r=0$, show that

$$\phi(r,\theta) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{a}\displaystyle\sum_{l=0}^{\infty}\alpha_l\left(\dfrac{r}{a}\right)^l P_l(\cos\theta), & r<a, \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{r}\displaystyle\sum_{l=0}^{\infty}\beta_l\left(\dfrac{a}{r}\right)^l P_l(\cos\theta), & a<r. \end{cases} \tag{22}$$

(e) Using Eq. (22), we have

$$\phi(r,0) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{a}\displaystyle\sum_{l=0}^{\infty}\alpha_l\left(\dfrac{r}{a}\right)^l, & r<a, \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{r}\displaystyle\sum_{l=0}^{\infty}\beta_l\left(\dfrac{a}{r}\right)^l, & a<r. \end{cases} \tag{23}$$

where we used $P_l(1) = 1$. Comparing Eqs. (16) and (23) show that

$$\alpha_l = \begin{cases} 1 & l = 0, \\ -1 & l = 1, \\ 0 & l = 3, 5, 7, \ldots, \\ \dfrac{(-1)^{n-1}}{2n} \dfrac{[2(n-1)]!}{2^{2(n-1)}[(n-1)!]^2}, & l = 2n, \quad n = 1, 2, 3, \ldots, \end{cases} \tag{24}$$

and

$$\beta_l = \begin{cases} 0 & l = 1, 3, 5, \ldots, \\ \dfrac{(-1)^n}{2(n+1)} \dfrac{(2n)!}{2^{2n}(n!)^2} & l = 2n, \quad n = 0, 1, 2, 3, \ldots. \end{cases} \tag{25}$$

Thus, show that

$$\phi(r,\theta) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0} \dfrac{2Q}{a} \left[1 - \dfrac{r}{a} P_1(\cos\theta) + \displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{2n} \dfrac{[2(n-1)]!}{2^{2(n-1)}[(n-1)!]^2} \left(\dfrac{r}{a}\right)^{2n} P_{2n}(\cos\theta) \right], & r < a \\ \dfrac{1}{4\pi\varepsilon_0} \dfrac{2Q}{r} \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n}{2(n+1)} \dfrac{(2n)!}{2^{2n}(n!)^2} \left(\dfrac{a}{r}\right)^{2n} P_{2n}(\cos\theta), & a < r. \end{cases} \tag{26}$$

(f) For $r \ll a$ the disc should simulate a plate of infinite extent. Show that

$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_0} \frac{2Q}{a} \left[1 - \frac{z}{a} \right] + \mathcal{O}\left(\frac{z}{a}\right)^2, \tag{27}$$

using $r P_1(\cos\theta) = z$. This leads to the electric field for a plate of infinite extent,

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi = \hat{\mathbf{z}} \frac{\sigma}{2\varepsilon_0}, \tag{28}$$

where $\sigma = Q/(\pi a^2)$.