Homework No. 03 (Fall 2020)

PHYS 520A: ELECTROMAGNETIC THEORY I

Department of Physics, Southern Illinois University–Carbondale

Due date: Friday, 2020 Sep 11, 11.00am

1. (**50 points.**) The Maxwell equations, in SI units, are

$$\boldsymbol{\nabla}\cdot\mathbf{D} = \rho, \tag{1a}$$
$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0, \tag{1b}$$
$$-\boldsymbol{\nabla}\times\mathbf{E} - \frac{\partial}{\partial t}\mathbf{B} = 0, \tag{1c}$$
$$\boldsymbol{\nabla}\times\mathbf{H} - \frac{\partial}{\partial t}\mathbf{D} = \mathbf{J}, \tag{1d}$$

where

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}, \tag{2a}$$
$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}. \tag{2b}$$

The Lorentz force, in SI units, is

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}. \tag{3}$$

We have

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}. \tag{4}$$

The above quantities will be addressed with subscripts SI in the following. The corresponding quantities in Gaussian (G) units and Heaviside-Lorentz (HL) units are obtained using the conversions

$$\sqrt{\frac{\varepsilon_0}{4\pi}}\,\mathbf{D}_{\text{G}} = \mathbf{D}_{\text{SI}} = \sqrt{\varepsilon_0}\,\mathbf{D}_{\text{HL}}, \qquad \sqrt{4\pi\varepsilon_0}\,\rho_{\text{G}} = \rho_{\text{SI}} = \sqrt{\varepsilon_0}\,\rho_{\text{HL}}, \tag{5a}$$
$$\frac{1}{\sqrt{4\pi\varepsilon_0}}\,\mathbf{E}_{\text{G}} = \mathbf{E}_{\text{SI}} = \frac{1}{\sqrt{\varepsilon_0}}\,\mathbf{E}_{\text{HL}}, \qquad \sqrt{4\pi\varepsilon_0}\,\mathbf{P}_{\text{G}} = \mathbf{P}_{\text{SI}} = \sqrt{\varepsilon_0}\,\mathbf{P}_{\text{HL}}, \tag{5b}$$
$$\frac{1}{\sqrt{4\pi\mu_0}}\,\mathbf{H}_{\text{G}} = \mathbf{H}_{\text{SI}} = \frac{1}{\sqrt{\mu_0}}\,\mathbf{H}_{\text{HL}}, \qquad \sqrt{4\pi\varepsilon_0}\,\mathbf{J}_{\text{G}} = \mathbf{J}_{\text{SI}} = \sqrt{\varepsilon_0}\,\mathbf{J}_{\text{HL}}, \tag{5c}$$
$$\sqrt{\frac{\mu_0}{4\pi}}\,\mathbf{B}_{\text{G}} = \mathbf{B}_{\text{SI}} = \sqrt{\mu_0}\,\mathbf{B}_{\text{HL}}, \qquad \sqrt{\frac{4\pi}{\mu_0}}\mathbf{M}_{\text{G}} = \mathbf{M}_{\text{SI}} = \frac{1}{\sqrt{\mu_0}}\mathbf{M}_{\text{HL}}. \tag{5d}$$

Note that the Heaviside-Lorentz units are obtained from Gaussian units by dropping the 4π's, which is called rationalization in this context.

(a) Starting from the Maxwell equations and Lorentz force in SI units, derive the corresponding equations in Gaussian units. The Maxwell equations, in Gaussian units, are

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{D}_{\mathrm{G}} &= 4\pi \rho_{\mathrm{G}}, && \text{(6a)} \\
\boldsymbol{\nabla} \cdot \mathbf{B}_{\mathrm{G}} &= 0, && \text{(6b)} \\
-\boldsymbol{\nabla} \times \mathbf{E}_{\mathrm{G}} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}_{\mathrm{G}} &= 0, && \text{(6c)} \\
\boldsymbol{\nabla} \times \mathbf{H}_{\mathrm{G}} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{D}_{\mathrm{G}} &= \frac{4\pi}{c}\mathbf{J}_{\mathrm{G}}, && \text{(6d)}
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{D}_{\mathrm{G}} &= \mathbf{E}_{\mathrm{G}} + 4\pi \mathbf{P}_{\mathrm{G}}, && \text{(7a)} \\
\mathbf{H}_{\mathrm{G}} &= \mathbf{B}_{\mathrm{G}} - 4\pi \mathbf{M}_{\mathrm{G}}. && \text{(7b)}
\end{aligned}$$

The Lorentz force, in Gaussian units, is

$$\mathbf{F} = q_{\mathrm{G}}\mathbf{E}_{\mathrm{G}} + q_{\mathrm{G}}\frac{\mathbf{v}}{c} \times \mathbf{B}_{\mathrm{G}}. \tag{8}$$

Here charge q_{G} has the same conversion as charge density ρ_{G}.

(b) Starting from the Maxwell equations and Lorentz force in SI units, derive the corresponding equations in Lorentz-Heaviside units. The Maxwell equations, in Heaviside-Lorentz units, are

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{D}_{\mathrm{HL}} &= \rho_{\mathrm{HL}}, && \text{(9a)} \\
\boldsymbol{\nabla} \cdot \mathbf{B}_{\mathrm{HL}} &= 0, && \text{(9b)} \\
-\boldsymbol{\nabla} \times \mathbf{E}_{\mathrm{HL}} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}_{\mathrm{HL}} &= 0, && \text{(9c)} \\
\boldsymbol{\nabla} \times \mathbf{H}_{\mathrm{HL}} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{D}_{\mathrm{HL}} &= \frac{1}{c}\mathbf{J}_{\mathrm{HL}}, && \text{(9d)}
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{D}_{\mathrm{HL}} &= \mathbf{E}_{\mathrm{HL}} + \mathbf{P}_{\mathrm{HL}}, && \text{(10a)} \\
\mathbf{H}_{\mathrm{HL}} &= \mathbf{B}_{\mathrm{HL}} - \mathbf{M}_{\mathrm{HL}}. && \text{(10b)}
\end{aligned}$$

The Lorentz force, in Heaviside-Lorentz units, is

$$\mathbf{F} = q_{\mathrm{HL}}\mathbf{E}_{\mathrm{HL}} + q_{\mathrm{HL}}\frac{\mathbf{v}}{c} \times \mathbf{B}_{\mathrm{HL}}. \tag{11}$$

Here charge q_{HL} has the same conversion as charge density ρ_{HL}.

2. (**20 points.**) In Gaussian units the power radiated by an accelerated charged particle of charge e is given by the Larmor formula,

$$P = \frac{2\,e^2}{3\,c^3}\,a^2, \tag{12}$$

where a is the acceleration of the charged particle. Write down the Larmor formula in SI units, and in Lorentz-Heaviside units.

3. (**30 points.**) The fine-structure constant, in Gaussian units,

$$\alpha = \frac{e^2}{\hbar c}, \tag{13}$$

is the parameter that characterizes the strength of the electromagnetic interaction.

   (a) Write down the corresponding expression for fine-structure constant in SI units, and in Lorentz-Heaviside units.

   (b) Verify that the fine-structure constant is a dimensionless quantity. Show that the numerical value of the fine-structure constant is independent of the system of units.

   (c) Evaluate the numerical value for the reciprocal of the fine-structure constant, α^{-1}. (A periodic table based on quantum electrodynamics breaks down for atomic numbers greater than α^{-1}.)

4. (**30 points.**) The relation between charge density and current density,

$$\frac{\partial}{\partial t}\rho(\mathbf{r},t) + \boldsymbol{\nabla}\cdot\mathbf{j}(\mathbf{r},t) = 0, \tag{14}$$

is the general statement of the conservation of charge.

   (a) Derive the statement of conservation of charge in Eq. (14) from the Maxwell equations.
   Hint: Take time derivative of Gauss's law and divergence of Ampere's law.

   (b) For an arbitrarily moving point particle with charge q, the charge and current densities are

$$\rho(\mathbf{r},t) = q\delta^{(3)}(\mathbf{r} - \mathbf{r}_a(t)) \tag{15}$$

   and

$$\mathbf{j}(\mathbf{r},t) = q\mathbf{v}_a(t)\,\delta^{(3)}(\mathbf{r} - \mathbf{r}_a(t)), \tag{16}$$

   where $\mathbf{r}_a(t)$ is the position vector and

$$\mathbf{v}_a(t) = \frac{d\mathbf{r}_a}{dt} \tag{17}$$

   is the velocity of the charged particle. Verify the statement of the conservation of charge in Eq. (14) for a point particle.