Notes on (calculus based) Physics

Prachi Parashar[1] and K. V. Shajesh[2]

Department of Physics,
Southern Illinois University–Carbondale,
Carbondale, Illinois 62901, USA.

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These are notes prepared for the benefit of students enrolled in PHYS-205A and PHYS-205B, calculus based introductory physics courses for non-physics majors, at Southern Illinois University–Carbondale. The following textbooks were extensively used in this compilation.

1. (Assigned Textbook:)
   Physics: for scientists and engineers with modern physics, Ninth Edition,
   Raymond A. Serway and John W. Jewett, Jr.,
   Brooks/Cole Cengage Learning.

2. Fundamentals of physics, Fifth Edition,
   David Halliday, Robert Resnick, and Jearl Walker,
   John Wiley & Sons, Inc.

These notes were primarily written in Fall 2015. It will be updated periodically, and will evolve during the semester. It is not a substitute for the assigned textbook for the course, but a supplement prepared as a study-guide.

[1]EMAIL: prachi@nhn.ou.edu
[2]EMAIL: kvshajesh@gmail.com, URL: http://www.physics.siu.edu/~shajesh

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Part I

Mechanics

Chapter 1

Measurement

1.1 International System (SI) of units

Three of the total of seven SI base units are

Physical Quantity	Dimension	Unit Name	Unit Symbol
Time	T	second	s
Length	L	meter	m
Mass	M	kilogram	kg

The remaining four physical quantities in the SI base units are: charge (measured in Coulomb), temperature (measured in Kelvin), amount of substance (measured in mole), and luminosity (measured in candela).

Orders of magnitude of physical quantities are written in powers of ten using the following prefixes:

$$\mathrm{c} = 10^{-2}, \qquad \mathrm{m} = 10^{-3}, \qquad \mu = 10^{-6}, \qquad \mathrm{n} = 10^{-9}, \qquad \mathrm{p} = 10^{-12}, \tag{1.1a}$$

$$\mathrm{d} = 10^{2}, \qquad \mathrm{k} = 10^{3}, \qquad \mathrm{M} = 10^{6}, \qquad \mathrm{G} = 10^{9}, \qquad \mathrm{T} = 10^{12}. \tag{1.1b}$$

Lecture-Example 1.1: (Serway and Jewett, 9 ed.)
Why is the following situation impossible? A room measures 4.0 m × 4.0 m, and its ceiling is 3.0 m high. A person completely wallpapers the walls of the room with the pages of a book which has 1000 pages of text (on 500 sheets) measuring 0.21 m× 0.28 m. The person even covers the door and window.

1.2 Dimensional analysis

Addition and subtraction is performed on similar physical quantities. Consider the mathematical relation between distance x, time t, velocity v, and acceleration a, given by

$$x = vt + \frac{1}{2}at^2. \tag{1.2}$$

This implies that

$$\big[x\big] = \big[vt\big] = \big[at^2\big] = L, \tag{1.3}$$

where we used the notation involving the square brackets

$$\big[a\big] = \text{dimension of the physical quantity represented by the symbol } a. \tag{1.4}$$

10^{-35} m	Planck length
10^{-18} m	size of electron
10^{-15} m	size of proton
10^{-10} m	size of atom
10^{-8} m	size of a virus
10^{-6} m	size of a bacteria
10^{0} m	size of a human
10^{6} m	size of Earth
10^{12} m	size of solar system
10^{15} m	distance to closest star
10^{21} m	size of a galaxy
10^{24} m	distance to closest galaxy
10^{25} m	size of observable universe

Table 1.1: Orders of magnitude (length). See also a slideshow titled Secret Worlds: The Universe Within, which depicts the relative scale of the universe.

Mathematical functions, like logarithm and exponential, are evaluated on numbers, which are dimensionless.

Lecture-Example 1.2:
Consider the mathematical expression

$$x = vt + \frac{1}{2!}at^2 + \frac{1}{3!}bt^3 + \frac{1}{4!}ct^4, \tag{1.5}$$

where x is measured in units of distance and t is measured in units of time. Determine the dimension of the physical quantities represented by the symbols v, a, b, and c.

- Deduce $\big[x\big] = \big[vt\big]$. Thus, we have $\big[v\big] = LT^{-1}$.
- Deduce $\big[x\big] = \big[at^2\big]$. Thus, we have $\big[a\big] = LT^{-2}$.
- Deduce $\big[x\big] = \big[bt^3\big]$. Thus, we have $\big[b\big] = LT^{-3}$.
- Deduce $\big[x\big] = \big[ct^4\big]$. Thus, we have $\big[c\big] = LT^{-4}$.

Lecture-Example 1.3: (Wave equation)
Consider the mathematical expression, for a travelling wave,

$$y = A\cos(kx - \omega t + \delta), \tag{1.6}$$

where x and y are measured in units of distance, t is measured in units of time, and δ is measured in units of angle (radians, that is dimensionless). Deduce the dimensions of the physical quantities represented by the symbols A, k, and ω. Further, what can we conclude about the nature of physical quantity constructed by $\frac{\omega}{k}$?

- Deduce $\big[y\big] = \big[A\big]$. Thus, conclude $\big[A\big] = L$.
- Deduce $\big[kx\big] = \big[\delta\big] = 1$. Thus, conclude $\big[k\big] = L^{-1}$.
- Deduce $\big[\omega t\big] = \big[\delta\big] = 1$. Thus, conclude $\big[\omega\big] = T^{-1}$.

- Deduce $\big[kx\big] = \big[\omega t\big]$. Thus, conclude that $\big[\frac{\omega}{k}\big] = LT^{-1}$. This suggests that the construction $\frac{\omega}{k}$ measures speed.

Lecture-Example 1.4: (Weyl expansion)
The list of overtones (frequencies of vibrations) of a drum is completely determined by the shape of the drumhead. Is the converse true? That is, what physical quantities regarding the shape of a drum can one infer, if the complete list of overtones is given. This is popularly stated as 'Can one hear the shape of a drum?' Weyl expansion, that addresses this question, is

$$E = \frac{A}{\delta^3} + \frac{C}{\delta^2} + \frac{B}{\delta} + a_0 + a_1\delta + a_2\delta^2 + \ldots, \tag{1.7}$$

where E is measured in units of inverse length, and δ is measured in units of length. Deduce that the physical quantities A and C have the dimensions of area and circumference, respectively.

Lecture-Example 1.5:
What can you deduce about the physical quantity c in the famous equation

$$E = mc^2, \tag{1.8}$$

if the energy E has the dimensions ML^2T^{-2} and mass m has the dimension M.

- $\big[c\big] = LT^{-1}$. Thus, the physical quantity c has the dimension of speed.

1.3 Homework problems

Homework-Problem 1.1: Two spheres are cut from a material of uniform density. One has radius 5.00 cm. The mass of the other is eight times greater. Find its radius.

Hints:

- Uniform density implies constant mass per unit volume. Thus, deduce the ratio

$$\frac{M_1}{R_1^3} = \frac{M_2}{R_2^3}. \tag{1.9}$$

Homework-Problem 1.2: The figure below shows a frustum of a cone.
Match each of the expressions,

$$\pi(r_1 + r_2)\sqrt{h^2 + (r_2 - r_1)^2}, \tag{1.10a}$$
$$2\pi(r_1 + r_2), \tag{1.10b}$$
$$\frac{1}{3}\pi h(r_1^2 + r_1 r_2 + r_2^2), \tag{1.10c}$$

with the quantity it describes: the total circumference of the flat circular faces, the volume, and the are of the curved surface.

Hints:

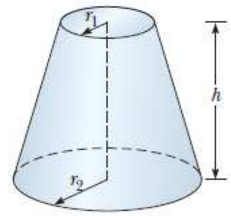

Figure 1.1: Homework-Problem 1.2.

- Determine the dimension of each expression and compare it to the dimension of circumference, area, and volume.

Homework-Problem 1.3: Newton's law of universal gravitation is represented by

$$F = \frac{GMm}{r^2} \tag{1.11}$$

where F is the magnitude of the gravitational force exerted by one small object on another, M and m are the masses of the objects, and r is a distance. Force has the SI units kg m/s^2. What are the SI units of the proportionality constant G?

Hints:

- Use
$$\big[F\big] = \frac{\big[G\big]\big[M\big]\big[m\big]}{\big[r\big]^2}, \tag{1.12}$$
and $\big[F\big] = MLT^{-2}$, $\big[M\big] = \big[m\big] = M$, $\big[r\big] = L$.

Homework-Problem 1.4: Assume the equation $x = At^3 + Bt$ describes the motion of a particular object, with x having the dimension of length and t having the dimension of time. Determine the dimensions of the constants A and B. Further, determine the dimensions of the derivative $\frac{dx}{dt} = 3At^2 + B$.

Hints:

- Deduce and use $\big[x\big] = \big[At^3\big] = \big[Bt\big]$.
- Deduce and use $\big[\frac{dx}{dt}\big] = \big[At^2\big] = \big[B\big]$.

Homework-Problem 1.5: Suppose your hair grows at the rate $\frac{1}{32}$ inch per day. Find the rate at which it grows in nanometers per second. Because the distance between atoms in a molecule is on the order of 0.1 nm, your answer suggests how rapidly layers of atoms are assembled in this protein synthesis.

Hints:

- Convert units,

$$\frac{1}{32}\frac{\text{inch}}{\text{day}} = \ ? \ \frac{\text{nm}}{\text{s}}. \tag{1.13}$$

Homework-Problem 1.6: The distance from the Sun to the nearest star is about 4×10^{16} m. The Milky Way galaxy is roughly a disk of diameter $\sim 10^{21}$ m and thickness $\sim 10^{19}$ m. Find the order of magnitude of the number of stars in the Milky Way. Assume the distance between the Sun and our nearest neighbor is typical.

Hints:

- Find volume of Milky Way, using volume of disc $\pi R^2 h$. Let the volume occupied per star be a cube of side 4×10^{16} m. Divide the two volumes to estimate the number of stars in Milky Way.

Chapter 2

Motion in one dimension

2.1 Motion

The pursuit of science is to gain a fundamental understanding of the principles governing our nature. A fundamental understanding includes the ability to make predictions.

Time

The very idea of prediction stems from the fact that time t always moves forward, that is,

$$\Delta t = t_f - t_i > 0, \tag{2.1}$$

where t_i is an initial time and t_f represents a time in the future. We will often choose the initial time $t_i = 0$.

Position

Our immediate interest would be to predict the position of an object. The position of an object (in space), relative to another point, is unambiguously specified as a vector $\vec{\mathbf{x}}$. The position is a function of time, that is, $\vec{\mathbf{x}}(t)$. Newtonian mechanics, the subject of discussion, proposes a strategy to determine the function $\vec{\mathbf{x}}(t)$, thus offering to predict the position of the object in a future time. This sort of prediction is exemplified every time a spacecraft is sent out, because we predict that it will be at a specific point in space at a specific time in the future. We will mostly be interested in the change in position,

$$\Delta\vec{\mathbf{x}} = \vec{\mathbf{x}}(t_f) - \vec{\mathbf{x}}(t_i). \tag{2.2}$$

Velocity

The instantaneous velocity of an object at time t is defined as the ratio of the change in position and change in time, which is unambiguous in the instantaneous limit,

$$\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{x}}}{dt} = \lim_{\Delta t\to 0} \frac{\Delta\vec{\mathbf{x}}}{\Delta t}. \tag{2.3}$$

We recognize the instantaneous velocity as the derivative of the position with respect to time. The magnitude of the instantaneous velocity vector is defined as the speed.

The average velocity is defined as

$$\vec{\mathbf{v}}_{\text{avg}} = \frac{\vec{\mathbf{x}}_f - \vec{\mathbf{x}}_i}{t_f - t_i} = \frac{1}{t_f - t_i}\int_{t_i}^{t_f} dt\,\vec{\mathbf{v}}(t), \tag{2.4}$$

where we used the fact that integration is the anti of derivative. This average velocity, defined using the first equality in Eq. (2.4), is overly used in non-calculus-based discussions on the topic, which has its limitations, but is nevertheless sufficient for a requisite understanding. In this context the speed is often also associated with the magnitude of the average velocity.

Lecture-Example 2.1: (Case $t_1 = t_2$)
While travelling in a straight line a car travels the first segment of distance d_1 in time t_1 at an average velocity v_1, and it travels the second segment of distance d_2 in time $t_2 = t_1$ at an average velocity v_2. Show that the velocity of the total trip is given by the average of the individual velocities,

$$v_{\text{tot}} = \frac{v_1 + v_2}{2}. \tag{2.5}$$

Lecture-Example 2.2: (Case $d_1 = d_2$)
While travelling in a straight line a car travels the first segment of distance d_1 in time t_1 at an average velocity v_1, and it travels the second segment of distance $d_2 = d_1$ in time t_2 at an average velocity v_2. Show that inverse of the velocity of the total trip is given by the average of the inverse of the individual velocities,

$$\frac{2}{v_{\text{tot}}} = \frac{1}{v_1} + \frac{1}{v_2}. \tag{2.6}$$

- Consider the following related example. You travel the first half segment of a trip at an average velocity of 50 miles/hour. What is the average velocity you should maintain during the second segment, of equal distance, to login an average velocity of 60 miles/hour for the total trip? Repeat for the case when you travel the first segment at 45 miles/hour.
- Next, repeat for the case when you travel the first segment at 30 miles/hour. Comprehend this. (Hint: Assume the total distance to be 60 miles and calculate the time remaining for the second segment.)

Acceleration

The acceleration of an object at time t is defined as the rate of change in velocity,

$$\vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{v}}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\mathbf{v}}}{\Delta t} = \frac{d^2\vec{\mathbf{x}}}{dt^2}. \tag{2.7}$$

Acceleration is the second derivative of position with respect to time.

Lecture-Example 2.3: A particle's position is given by

$$x(t) = v_0\, t + \frac{1}{2}\, a_0\, t^2 + \frac{1}{6}\, b_0\, t^3. \tag{2.8}$$

- Determine the particle's velocity as a function of time.
- What is the particle's velocity at time $t = \frac{v_0}{a_0}$?
- Determine the particle's acceleration as a function of time.
- What is the particle's acceleration at time $t = \frac{a_0}{b_0}$?

Lecture-Example 2.4: (Review of maxima-minima)
A ball is thrown vertically upward from the top of a 10.0 m high building at a speed of 20.0 m/s. Starting the clock right when the ball is thrown, the height of the ball (measured from the ground) as a function of time is given by

$$y(t) = 10.0 + 20.0t - 5t^2. \tag{2.9}$$

- Plot $y(t)$.
- Determine the time taken by the ball to reach the maximum height.
- Determine the maximum height of the ball.
- Determine the time taken for the ball to reach the ground. This involves finding the solution of a quadratic equation. Give a physical interpretation for the both the solutions.

Lecture-Example 2.5: Starting at time $t = 0$, an object moves along a straight line. Its coordinate in meters is given by

$$x(t) = 75\,t - 1.0\,t^3, \tag{2.10}$$

where t is in seconds. What is its acceleration when it momentarily stops?

2.2 Graphical analysis

In calculus we learn that the slope of the tangent to the function is the derivative, and the area under the function is the integral. These ideas lead to the following interpretations.

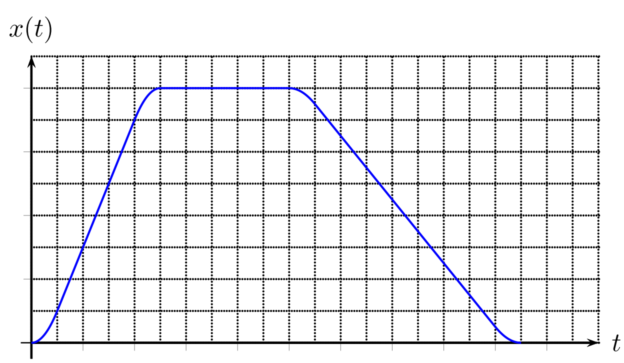

Figure 2.1: A position-time graph.

Position-time graph

In the position-time graph the slope of the tangent to the position curve at a certain time represents the instantaneous velocity. The inverse of the curvature of the position curve at a certain time is related to the instantaneous acceleration.

Velocity-time graph

In the velocity-time graph the slope of the tangent to the velocity curve at a certain time represents the instantaneous acceleration. The area under the velocity curve is the position up to a constant.

Acceleration-time graph

The area under the acceleration curve is the velocity up to a constant.

2.3 Motion with constant acceleration

Definition of velocity and acceleration supplies the two independent equations for the case of constant acceleration:

$$\frac{\Delta x}{\Delta t} = \frac{v_f + v_i}{2}, \tag{2.11a}$$

$$a = \frac{v_f - v_i}{\Delta t}. \tag{2.11b}$$

It is worth emphasizing that the relation in Eq. (2.11a) is valid only for the case of constant velocity. It is obtained by realizing that velocity is a linear function of time for constant acceleration in Eq. (2.4). Eqs. (2.11a) and (2.11b) are two independent equations involving five independent variables: $\Delta t, \Delta x, v_i, v_f, a$. We can further deduce,

$$\Delta x = v_i \Delta t + \frac{1}{2} a \Delta t^2, \tag{2.11c}$$

$$\Delta x = v_f \Delta t - \frac{1}{2} a \Delta t^2, \tag{2.11d}$$

$$v_f^2 = v_i^2 + 2a\Delta x, \tag{2.11e}$$

obtained by subtracting, adding, and multiplying, Eqs. (2.11a) and (2.11b), respectively. There is one of the five variables missing in each of the Eqs. (2.11), and it is usually the variable missing in the discussion in a particular context.

Lecture-Example 2.6: While driving on a highway you press on the gas pedals for 20.0 seconds to increase your speed from an initial speed of 40.0 miles/hour to a final final speed of 70.0 miles/hour. Assuming uniform acceleration find the acceleration.
Answer: $0.67\,\text{m/s}^2$.

To gain an intuitive feel for the magnitude of the velocities it is convenient to observe that, using 1 mile $\sim$ 1609 m,

$$2\,\frac{\text{miles}}{\text{hour}} \sim 1\,\frac{\text{m}}{\text{s}}, \tag{2.12}$$

correct to one significant digit, which is more accurately 1 miles/hour=0.447 m/s.

Lecture-Example 2.7:
While standing on a $h = 50.0\,\text{m}$ tall building you throw a stone straight upwards at a speed of $v_i = 15\,\text{m/s}$.

- How long does the stone take to reach the ground. (Be careful with the relative signs for the variables.) Mathematically this leads to two solution. Interpret the negative solution.
- How high above the building does the stone reach?

1 m/s	human walking speed
10 − 50 m/s	typical speed on a highway
340 m/s	speed of sound, speed of a typical fighter jet
1000 m/s	speed of a bullet
11 200 m/s	minimum speed necessary to escape Earth's gravity
299 792 458 m/s	speed of light

Table 2.1: Orders of magnitude (speed)

$1\,\text{m/s}^2$	typical acceleration on a highway
$g = 9.8\,\text{m/s}^2$	acceleration due to gravity on surface of Earth
$3g$	space shuttle launch
$5g$	causes dizziness (and fear) in humans
$6g$	high-g roller coasters and dragsters
$8g$	fighter jets pulling out of a dive
$20g$	damage to capillaries
$50g$	causes death, a typical car crash

Table 2.2: Orders of magnitude (acceleration)

- What is the velocity of the stone right before it reaches the ground?
- How will your results differ if the stone was thrown vertically downward with the same speed?

Lecture-Example 2.8:
The kinematic equations are independent of mass. Thus. the time taken to fall a certain distance is independent of mass. The following BBC video captures the motion of a feather and a bowling ball when dropped together inside the world's biggest vacuum chamber.

https://www.youtube.com/watch?v=E43-CfukEgs

Lecture-Example 2.9:
A fish is dropped by a pelican that is rising steadily at a speed $v_i = 4.0\,\text{m/s}$. Determine the time taken for the fish to reach the water 15.0 m below. How high above the water is the pelican when the fish reaches the water?

- The distance the fish falls is given by, (x_f is chosen to be positive upward so that v_i is positive when the fish is moving upward,) is given by,

$$x_f = v_i t - \frac{1}{2} g t^2, \tag{2.13}$$

and the distance the pelican moves up in the same time is given by (x_p is chosen to be positive upward)

$$x_p = v_i t. \tag{2.14}$$

At the time the fish hits the water we have $x_f = -15.0\,\text{m}$. (Answer: $t = 2.2\,\text{s}$ or $-1.4\,\text{s}$. Interpret the meaning of both solutions and chose the one appropriate to the context. Use this time to calculate $x_p = 8.8\,\text{m}$, which should be added to 15.0 m to determine how high above the water pelican is at this time.)

- Repeat for the case when the pelican is descending at a speed v_i. Compare the answers for the times with the negative solution in the rising case. (Answer: $t = 1.4\,\mathrm{s}$ or $-2.2\,\mathrm{s}$. Use this time to calculate $x_p = -5.6\,\mathrm{m}$.)

Lecture-Example 2.10: (Speeder and cop)
A speeding car is moving at a constant speed of $v = 80.0\,\mathrm{miles/hour}$ ($35.8\,\mathrm{m/s}$). A police car is initially at rest. As soon as the speeder crosses the police car the cop starts chasing the speeder at a constant acceleration of $a = 2.0\,\mathrm{m/s^2}$. Determine the time it takes for the cop to catch up with the speeder. Determine the distance travelled by the cop in this time.

- The distance travelled by the cop is given by

$$x_c = \frac{1}{2}at^2, \tag{2.15}$$

and the distance travelled by the speeder is given by

$$x_s = vt. \tag{2.16}$$

When the cop catches up with the speeder we have

$$x_s = x_c. \tag{2.17}$$

- How would your answers change if the cop started the chase $t_0 = 1.0\,\mathrm{s}$ after the speeder crossed the cop? This leads to two mathematically feasible solutions, interpret the unphysical solution. Plot the position of the speeder and the cop on the same position-time plot.

Lecture-Example 2.11:
A key falls from a bridge that is $50.0\,\mathrm{m}$ above the water. It falls directly into a boat that is moving with constant velocity v_b, that was $10.0\,\mathrm{m}$ from the point of impact when the key was released. What is the speed v_b of the boat?

- The distance the key falls is given by

$$d_k = \frac{1}{2}gt^2, \tag{2.18}$$

and the distance the boat moves in the same time is given by

$$d_b = v_b t. \tag{2.19}$$

Eliminating t gives a suitable equation.

Lecture-Example 2.12: (Drowsy cat)
A drowsy cat spots a flowerpot that sails first up and then down past an open window. The pot is in view for a total of $0.50\,\mathrm{s}$, and the top-to-bottom height of the window is $2.00\,\mathrm{m}$. How high above the window top does the flower pot go?

- The time taken to cross the window during the upward motion is the same as the time taken during the downward motion. Determine the velocity of the flowerpot as it crosses the top edge of the window, then using this information find the answer.
  Answer: 2.34 m

Lecture-Example 2.13:
A man drops a rock into a well. The man hears the sound of the splash $T = 2.40\,\mathrm{s}$ after he releases the rock from rest. The speed of sound in air (at the ambient temperature) is $v_0 = 336\,\mathrm{m/s}$. How far below the top of the well h is the surface of the water? If the travel time for the sound is ignored, what percentage error is introduced when the depth of the well is calculated?

- The time taken for the rock to reach the surface of water is

$$t_1 = \frac{2h}{g}, \tag{2.20}$$

  and the time taken for the sound to reach the man is given by

$$t_2 = \frac{h}{v_0}, \tag{2.21}$$

  and it is given that

$$t_1 + t_2 = T. \tag{2.22}$$

  This leads to a quadratic equation in h which has the solutions

$$h = \frac{v_0^2}{g}\left[\left(1 + \frac{gT}{v_0}\right) \pm \sqrt{1 + 2\frac{gT}{v_0}}\right]. \tag{2.23}$$

  Travel time for the sound being ignored corresponds to the limit $v_0 \to \infty$. The parameter $gT/v_0 \sim 0.07$ tells us that this limit will correspond to an error of about 7%.

- The correct solution corresponds to the one from the negative sign, $h = 26.4\,\mathrm{m}$. The other solution, $h = 24630\,\mathrm{m}$, corresponds to the case where the rock hits the surface of water in negative time, which is of course unphysical in our context. Visualize this by plotting the path of the rock as a parabola, which is intersected by the path of sound at two points.

Lecture-Example 2.14: (An imaginary tale: The story of $\sqrt{-1}$, by Paul J. Nahin)
Imagine that a man is running at his top speed v to catch a bus that is stopped at a traffic light. When he is still a distance d from the bus, the light changes and the bus starts to move away from the running man with a constant acceleration a.

- When will the man catch the bus? In terms of the time-scale of the man,

$$t_1 = \frac{d}{v}, \tag{2.24}$$

  and the time-scale of the bus,

$$t_2 = \sqrt{\frac{2d}{a}}, \tag{2.25}$$

derive the following expression for time, $x = 2t_1/t_2$,

$$t = t_1 \left[\frac{1 \pm \sqrt{1 - x^2}}{x} \right]. \tag{2.26}$$

Plot the distances traversed by the man and the bus on the same plot. What is the interpretation of the crossings in the plots?

- What is the minimum speed necessary for the man to catch the bus?
- If we suppose that the man does not catch the bus, at what time is the man closest to the bus?

2.4 Homework problems

Homework-Problem 2.8: A hockey player '2' is standing on his skates on a frozen pond when an opposing player '1', moving with a uniform speed of $v_1 = 2.0\,\mathrm{m/s}$, skates by with the puck. After 1.00 s, the first player makes up his mind to chase his opponent. If he accelerates uniformly at $a_2 = 0.18\,\mathrm{m/s^2}$, determine each of the following.

1. How long does it take him to catch his opponent? (Assume the player with the puck remains in motion at constant speed.)
2. How far has he traveled in that time?

Hints:
The distance travelled by the first player is given by

$$x_1 = v_1 t_1, \tag{2.27}$$

and the distance travelled by the second player is given by

$$x_2 = \frac{1}{2} a_2 t_2^2. \tag{2.28}$$

We further have $t_2 = t_1 - 1.0$ and when catch up $x_1 = x_2$.

Homework-Problem 2.11: A ball '1' is thrown upward from the ground with an initial speed of $v_1 = 24.6\,\mathrm{m/s}$; at the same instant, another ball '2' is dropped from a building $H = 18\,\mathrm{m}$ high. After how long will the balls be at the same height above the ground?

Hints:
Deduce that the ball '1' climbs a distance y_1 given by

$$y_1 = v_1 t - \frac{1}{2} g t^2, \tag{2.29}$$

and the ball '2' falls down a distance y_2 given by

$$y_2 = \frac{1}{2} g t^2. \tag{2.30}$$

When the balls are at the same height we have

$$H = y_1 + y_2. \tag{2.31}$$

Chapter 3

Vector algebra

3.1 Vector

The position of an object on a plane, relative to an origin, is uniquely specified by the Cartesian coordinates (x, y), or the polar coordinates $(r; \theta)$. The position vector is mathematically expressed in the form

$$\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}, \tag{3.1}$$

where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are orthogonal unit vectors. The position vector is intuitively described in terms of its magnitude r and direction θ. These quantities are related to each other by the geometry of a right triangle,

$$r = \sqrt{x^2 + y^2}, \qquad x = r\cos\theta, \tag{3.2a}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right), \qquad y = r\sin\theta. \tag{3.2b}$$

A vector $\vec{\mathbf{A}}$, representing some physical quantity other than the position vector, will be mathematically represented by

$$\vec{\mathbf{A}} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}}, \tag{3.3}$$

whose magnitude will be represented by $|\vec{\mathbf{A}}|$ and the direction by the angle θ_A.

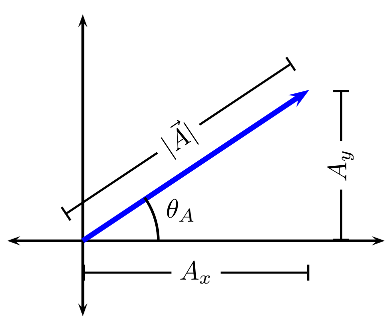

Figure 3.1: The right triangle geometry of a vector $\vec{\mathbf{A}}$.

Lecture-Example 3.1:
Find the components of a vector $\vec{\mathbf{A}}$ whose magnitude is 20.0 m and its direction is 30.0° counterclockwise with

respect to the positive x-axis.
Answer: $\vec{\mathbf{A}} = (17.3\,\hat{\mathbf{i}} + 10.0\,\hat{\mathbf{j}})\,\text{m}$.

Lecture-Example 3.2: (Caution)
Inverse tangent is many valued. In particular,

$$\tan\theta = \tan(\pi + \theta). \tag{3.4}$$

This leads to the ambiguity that the vectors, $\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ and $\vec{\mathbf{r}} = -x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$, produce the same direction θ using the formula $\tan^{-1}(y/x)$. This should be avoided by visually judging on the angles based on the quadrants the vector are in. Find the direction of the following two vectors:

$$\vec{\mathbf{A}} = 5.0\,\hat{\mathbf{i}} + 10\,\hat{\mathbf{j}}, \tag{3.5a}$$
$$\vec{\mathbf{B}} = -5.0\,\hat{\mathbf{i}} - 10\,\hat{\mathbf{j}}. \tag{3.5b}$$

We determine $\tan^{-1}(10/5) = \tan^{-1}(-10/-5) = 63°$. Since the vector $\vec{\mathbf{A}}$ is in the first quadrant we conclude that it makes $63°$ counterclockwise w.r.t. $+x$ axis, and the vector $\vec{\mathbf{B}}$ being in the third quadrant makes $63°$ counterclockwise w.r.t. $-x$ axis or $243°$ counterclockwise w.r.t. $+x$ axis.

3.2 Addition and subtraction of vectors

Consider two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ given by

$$\vec{\mathbf{A}} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}}, \tag{3.6a}$$
$$\vec{\mathbf{B}} = B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}}. \tag{3.6b}$$

The sum of the two vectors, say $\vec{\mathbf{C}}$, is given by

$$\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}} = (A_x + B_x)\,\hat{\mathbf{i}} + (A_y + B_y)\,\hat{\mathbf{j}}. \tag{3.7}$$

The difference of the two vectors, say $\vec{\mathbf{D}}$, is given by

$$\vec{\mathbf{D}} = \vec{\mathbf{A}} - \vec{\mathbf{B}} = (A_x - B_x)\,\hat{\mathbf{i}} + (A_y - B_y)\,\hat{\mathbf{j}}. \tag{3.8}$$

It should be pointed out that the magnitudes and directions of a vector do not satisfy these simple rules. Thus, to add vectors, we express the vectors in their component form, perform the operations, and then revert back to the magnitude and direction of the resultant vector.

Lecture-Example 3.3: Given that vector $\vec{\mathbf{A}}$ has magnitude $A = |\vec{\mathbf{A}}| = 15\,\text{m}$ and direction $\theta_A = 30.0°$ counterclockwise w.r.t x-axis, and that vector $\vec{\mathbf{B}}$ has magnitude $B = |\vec{\mathbf{B}}| = 20.0\,\text{m}$ and direction $\theta_B = 45.0°$ counterclockwise w.r.t x-axis. Determine the magnitude and direction of the sum of the vectors.

- The given vectors are determined to be

$$\vec{\mathbf{A}} = 13\,\hat{\mathbf{i}} + 7.5\,\hat{\mathbf{j}}, \tag{3.9a}$$
$$\vec{\mathbf{B}} = 14\,\hat{\mathbf{i}} + 14\,\hat{\mathbf{j}}. \tag{3.9b}$$

  We can show that

$$\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}} = (27\,\hat{\mathbf{i}} + 22\,\hat{\mathbf{j}})\,\text{m}. \tag{3.10}$$

The magnitude of vector $\vec{\mathbf{C}}$ is

$$C = |\vec{\mathbf{C}}| = \sqrt{27^2 + 22^2} = 35\,\mathrm{m}, \tag{3.11}$$

and its direction θ_C counterclockwise w.r.t. x-axis is

$$\theta_C = \tan^{-1}\left(\frac{22}{27}\right) = 39°. \tag{3.12}$$

3.3 Graphical method

Graphical method is based on the fact that the vector $\vec{\mathbf{A}} + \vec{\mathbf{B}}$ is diagonal of parallelogram formed by the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.

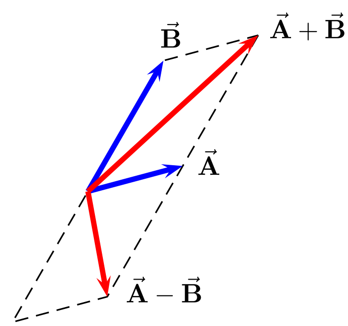

Figure 3.2: Graphical method for vector addition and subtraction.

Using the law of cosines,

$$C^2 = A^2 + B^2 - 2AB\cos\theta_{ab}, \tag{3.13}$$

and the law of sines,

$$\frac{A}{\sin\theta_{bc}} = \frac{B}{\sin\theta_{ca}} = \frac{C}{\sin\theta_{ab}}, \tag{3.14}$$

for a triangle, one determines the magnitude and direction of the sum of vectors.

Lecture-Example 3.4: (Caution)
Inverse sine function is many valued. In particular,

$$\sin\left(\frac{\pi}{2} - \theta\right) = \sin\left(\frac{\pi}{2} + \theta\right). \tag{3.15}$$

For example $\sin 85° = \sin 95° = 0.9962$. Consider the vector $\vec{\mathbf{A}}$ with magnitude $|\vec{\mathbf{A}}| = 1.0$ and direction $\theta_A = 0°$ w.r.t. $+x$ axis, and another vector $\vec{\mathbf{B}}$ with magnitude $|\vec{\mathbf{B}}| = 2.5$ and direction $\theta_B = 60°$ clockwise w.r.t. $-x$ axis. Using the law of cosines the magnitude of the vector $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ is determined as

$$C = \sqrt{1.0^2 + 2.5^2 - 2 \times 1.0 \times 2.5\cos 60} = 2.18. \tag{3.16}$$

Next, using the law of sines we find

$$\frac{2.18}{\sin 60} = \frac{2.5}{\sin\theta_C} \quad\rightarrow\quad \sin\theta_C = 0.993 \quad\rightarrow\quad \theta_C = 83.2°,\ 96.8°. \tag{3.17}$$

Settle this confusion by evaluating the angle between the vectors $\vec{\mathbf{B}}$ and $\vec{\mathbf{C}}$, and thus determine $\theta_C = 96.8°$.

Lecture-Example 3.5: (One Two Three… Infinity, by George Gamow)
"There was a young and adventurous man who found among his great-grandfather's papers a piece of parchment that revealed the location of a hidden treasure. The instructions read:

'Sail to ______ North latitude and ______ West longitude where thou wilt find a deserted island. There lieth a large meadow, not pent, on the north shore of the island where standeth a lonely oak and a lonely pine. There thou wilt see also an old gallows on which we once were wont to hang traitors. Start thou from the gallows and walk to the oak counting thy steps. At the oak thou must turn right by a right angle and take the same number of steps. Put here a spike in the ground. Now must thou return to the gallows and walk to the pine counting thy steps. At the pine thou must turn left by a right angle and see that thou takest the same number of steps, and put another spike in the ground. [Look] halfway between the spikes; the treasure is there.'

The instructions were quite clear and explicit, so our young man chartered a ship and sailed to the South Seas. He found the island, the field, the oak and the pine, but to his great sorrow the gallows was gone. Too long a time had passed since the document had been written; rain and sun and wind had disintegrated the wood and returned it to the soil, leaving no trace even of the place where it once had stood.

Our adventurous young man fell into despair, then in an angry frenzy began to [run] at random all over the field. But all his efforts were in vain; the island was too big! So he sailed back with empty hands. And the treasure is probably still there."

Show that one does not need the position of the gallows to find the treasure.

- Let the positions be oak tree: $\vec{\mathbf{A}}$, pine tree: $\vec{\mathbf{B}}$, gallows: $\vec{\mathbf{G}}$, spike A: $\vec{\mathbf{S}}_A$, spike B: $\vec{\mathbf{S}}_B$, treasure: $\vec{\mathbf{T}}$. Choose the origin at the center of the line segment connecting the oak tree and pine tree. Thus we can write

$$\vec{\mathbf{A}} = -d\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}}, \tag{3.18a}$$
$$\vec{\mathbf{B}} = d\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}}. \tag{3.18b}$$

In terms of the unknown position of the gallows,

$$\vec{\mathbf{G}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}, \tag{3.19}$$

show that

$$\vec{\mathbf{S}}_A = -(y+d)\,\hat{\mathbf{i}} + (x+d)\,\hat{\mathbf{j}}, \tag{3.20a}$$
$$\vec{\mathbf{S}}_B = (y+d)\,\hat{\mathbf{i}} - (x-d)\,\hat{\mathbf{j}}. \tag{3.20b}$$

Thus, find the position of the treasure,

$$\vec{\mathbf{T}} = \frac{1}{2}(\vec{\mathbf{S}}_A + \vec{\mathbf{S}}_B) = 0\,\hat{\mathbf{i}} + d\,\hat{\mathbf{j}}. \tag{3.21}$$

3.4 Homework problems

Homework-Problem 3.7: A man pushing a mop across a floor causes it to undergo two displacements. The first has a magnitude of 142 cm and makes an angle of 122° with $+x$ axis. The resultant displacement has a magnitude of 143 cm and is directed at an angle of 40.0° to $+x$ axis. Find the magnitude and direction of the second displacement.

- It is given that

$$\vec{\mathbf{A}} = -142\sin 32\,\hat{\mathbf{i}} + 142\cos 32\,\hat{\mathbf{j}}, \tag{3.22a}$$
$$\vec{\mathbf{B}} = B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}}, \tag{3.22b}$$
$$\vec{\mathbf{C}} = 143\cos 40\,\hat{\mathbf{i}} + 143\sin 40\,\hat{\mathbf{j}}, \tag{3.22c}$$

such that

$$\vec{\mathbf{A}} + \vec{\mathbf{B}} = \vec{\mathbf{C}}. \tag{3.23}$$

Show that $\vec{\mathbf{B}} = 184.8\,\hat{\mathbf{i}} - 28.5\,\hat{\mathbf{j}}$, and determine $|\vec{\mathbf{B}}| = 187\,\text{cm}$ and direction $\theta_B = 8.77^\circ$ clockwise w.r.t. $+x$ axis.

Chapter 4

Motion in two dimensions

4.1 Motion in 2D

Motion in each (orthogonal) direction is independently governed by the respective position, velocity, and acceleration, with time being common to all dimensions that links together. In terms of the position in each direction we can write the position vector as

$$\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}}. \tag{4.1}$$

The instantaneous velocity is defined as the rate of change of position, expressed as the derivative of the position, as

$$\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{r}}}{dt} = v_x(t)\,\hat{\mathbf{i}} + v_y(t)\,\hat{\mathbf{j}}. \tag{4.2}$$

The instantaneous acceleration is defined as the rate of change of velocity, expressed as the derivative of the velocity, as

$$\vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{v}}}{dt} = a_x(t)\,\hat{\mathbf{i}} + a_y(t)\,\hat{\mathbf{j}}. \tag{4.3}$$

Lecture-Example 4.1:
A particle is moving in the xy plane. Its initial position, at time $t = 0$, is given given by

$$\vec{\mathbf{r}}_0 = (2.0\,\hat{\mathbf{i}} + 3.0\,\hat{\mathbf{j}})\,\mathrm{m}, \tag{4.4}$$

and its initial velocity is given by

$$\vec{\mathbf{v}}_0 = (25\,\hat{\mathbf{i}} + 35\,\hat{\mathbf{j}})\ \frac{\mathrm{m}}{\mathrm{s}}. \tag{4.5}$$

Find the position and velocity of the particle at time $t = 15.0\,\mathrm{s}$ if it moves with uniform acceleration

$$\vec{\mathbf{a}} = (-1.0\,\hat{\mathbf{i}} - 10.0\,\hat{\mathbf{j}})\ \frac{\mathrm{m}}{\mathrm{s}^2}. \tag{4.6}$$

- The final position is determined using

$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}_0 + \vec{\mathbf{v}}_0\,t + \frac{1}{2}\vec{\mathbf{a}}\,t^2, \tag{4.7}$$

  and the final velocity is determined using

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}_0 + \vec{\mathbf{a}}\,t. \tag{4.8}$$

4.2 Projectile motion

Projectile motion is described by the uniform acceleration

$$\vec{\mathbf{a}} = 0\,\hat{\mathbf{i}} - g\,\hat{\mathbf{j}}, \tag{4.9}$$

where $g = 9.80\,\text{m/s}^2$ is the acceleration due to gravity.

Lecture-Example 4.2: (Maximum height of a projectile)
Show that the maximum height attained by a projectile is

$$H = \frac{v_0^2 \sin^2\theta_0}{2g}, \tag{4.10}$$

where v_0 is the magnitude of the initial velocity and it is projected at an angle θ_0.

Lecture-Example 4.3: (Range of a projectile)
Show that the range of a projectile is given by

$$R = \frac{v_0^2 \sin 2\theta_0}{g}, \tag{4.11}$$

where v_0 is the magnitude of the initial velocity and it is projected at an angle θ_0.

- Show that the range of a projectile is a maximum when it is projected at 45° with respect to horizontal.
- The fastest sprint speed recorded for a human is 12.4 m/s, (updated in 2015 Sep). If a person were to jump off with this speed in a long jump event, at an angle 45° with respect to the horizontal, he/she would cover a distance of 15.7 m. Instead, if a person were to jump off with this speed at an angle 20° with respect to the horizontal, he/she would cover a distance of 10.1 m. The world record for long jump is about 9 m. Apparently, the technique used by professional jumpers does not allow them to jump at 45° without compromising on their speed, they typically jump at 20°. This seems to suggest that there is room for clever techniques to be developed in long jump.
- Cheetah is the fastest land animal, about 30 m/s. They cover about 7 m in each stride. Estimate the angle of takeoff for each stride, assuming a simple model.
  Answer: 2°

Lecture-Example 4.4: (Half a parabola)
An airplane flying horizontally at a uniform speed of 40.0 m/s over level ground releases a bundle of food supplies. Ignore the effect of air on the bundle. The bundle is dropped from a height of 300.0 m.

- Observe that the initial vertical component of velocity of the bundle is zero, and the horizontal component of velocity remains constant.
- Determine the time taken for the drop. (Answer: 7.8 s.) Will this time change if the the airplane was moving faster or slower? Consider the extreme (unphysical) case when the airplane is horizontally at rest.
- Determine the horizontal distance covered by the bundle while it is in the air. (Answer: 313 m.)

- Determine the vertical and horizontal component of velocity just before it reaches the ground. (Answer: $\vec{\mathbf{v}}_f = (40.0\,\hat{\mathbf{i}} - 76\,\hat{\mathbf{j}})\,\mathrm{m/s}$.) Thus, determine the magnitude and direction of final velocity. (Answer: $|\vec{\mathbf{v}}_f| = 86\,\mathrm{m/s}$, $\theta_f = 62^\circ$ below the horizontal.)

Lecture-Example 4.5: (Baseball)
A batter hits a ball with an initial velocity $v_i = 30.0\,\mathrm{m/s}$ at an angle of 45° above the horizontal. The ball is $1.2\,\mathrm{m}$ above the ground at the time of hit. There is $10.0\,\mathrm{m}$ high fence, which is a horizontal distance $100.0\,\mathrm{m}$ away from the batter.

- Determine the horizontal and vertical components of the initial velocity. (Answer: $\vec{\mathbf{v}}_i = (21\,\hat{\mathbf{i}} + 21\,\hat{\mathbf{j}})\,\mathrm{m/s}$.)
- Determine the horizontal range of the ball, ignoring the presence of the fence. (Answer: $92\,\mathrm{m}$.)
- Determine the time the ball takes to traverse the horizontal distance to the fence. (Answer: $4.7\,\mathrm{s}$.)
- Determine the vertical distance of the ball when it reaches the fence. (Answer: $-9.1\,\mathrm{m}$.) Thus, analyze whether the ball clears the fence.
- Repeat the above analysis for $v_i = 32\,\mathrm{m/s}$. Does the ball clear the fence? What is the distance between the top of the fence and the center of the ball when the ball reaches the fence? (Answer: $y = 4.2\,\mathrm{m}$, implying the ball hits $5.8\,\mathrm{m}$ below the top of fence.)
- Repeat the above analysis for $v_i = 33\,\mathrm{m/s}$. Does the ball clear the fence? What is the distance between the top of the fence and the center of the ball when the ball reaches the fence? (Answer: $y = 1.0 \times 10^1\,\mathrm{m}$, up to two significant digits, implying the ball is right at the top of the fence. We can not conclude if it clears the fence accurately, without having more precise information.)

Lecture-Example 4.6: (Galileo's thought experiment, from Dialogue Concerning the Two Chief World Systems, translated by Stillman Drake)
Hang up a bottle that empties drop by drop into a vessel beneath it. Place this setup in a ship (or vehicle) moving with uniform speed. Will the drops still be caught in the vessel? What if the ship is accelerating?

Lecture-Example 4.7: (Bullseye)
A bullet is fired horizontally with speed $v_i = 400.0\,\mathrm{m/s}$ at the bullseye (from the same level). The bullseye is a horizontal distance $x = 100.0\,\mathrm{m}$ away.

Figure 4.1: Path of a bullet aimed at a bullseye.

- Since the bullet will fall under gravity, it will miss the bullseye. By what vertical distance does the bullet miss the bullseye? (Answer: $31\,\mathrm{cm}$.)
- At what angle above the horizontal should the bullet be fired to successfully hit the target? (Answer: 0.18°.)

Lecture-Example 4.8: (Simultaneously released target)
A bullet is aimed at a target (along the line). The target is released the instant the bullet is fired.

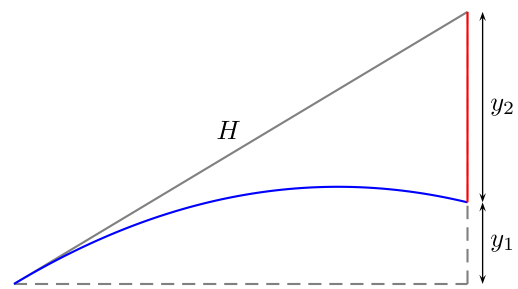

Figure 4.2: Path of the bullet (in blue) and path of the target (in red).

The path of the bullet is described by

$$y_1 = v_{iy}t - \frac{1}{2}gt^2, \tag{4.12}$$

and the path of the target is described by

$$y_2 = \frac{1}{2}gt^2. \tag{4.13}$$

Adding the two equations we have

$$H = \frac{(y_1 + y_2)}{\sin\theta_i} = v_i t, \tag{4.14}$$

which is the path of the bullet moving with uniform speed v_i along the hypotenuse H. Thus, the bullet does hit the target.

4.3 Centripetal acceleration

From the definition of acceleration,

$$\vec{\mathbf{a}} = \frac{d}{dt}\vec{\mathbf{v}}, \tag{4.15}$$

we can infer that uniform velocity implies zero acceleration. Here uniform means for constant with respect to time. Here we investigate the case when the magnitude of velocity, $v = |\vec{\mathbf{v}}|$, the speed, is uniform, but the direction of speed is not constant in time.

Uniform circular motion

A particle moving in a circle of radius R with uniform speed is termed uniform circular motion. Circular motion is periodic, so we introduce the time period T. A related quantity is the inverse of time period, the frequency,

$$f = \frac{1}{T}, \tag{4.16}$$

which is measured in units of revolutions per unit time, or more generally as number of times per unit time. Using the fact that

$$1\,\text{revolution} = 2\pi\,\text{radians} \tag{4.17}$$

we define the angular frequency

$$\omega = 2\pi f = \frac{2\pi}{T}. \tag{4.18}$$

Lecture-Example 4.9: A bus comes to a bus stop every 20 minutes. How frequently, in units of times per second, does the bus come to the bus stop? (Answer: 3 times/hour.)

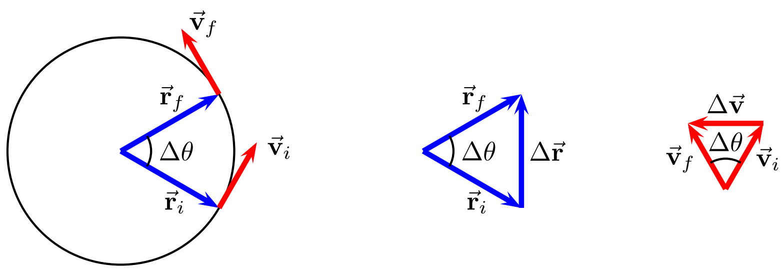

Figure 4.3: Change in position and velocity in uniform circular motion.

Magnitude of velocity in uniform circular motion

The angular frequency is the rate of change of angle θ per unit time. Thus, it is also called the angular velocity,

$$\omega = \frac{d\theta}{dt}. \tag{4.19}$$

The speed in uniform circular motion

$$v = \frac{2\pi R}{T} = \omega R. \tag{4.20}$$

Direction of velocity in uniform circular motion

Direction of velocity is decided by the direction of change in position $\Delta\vec{\mathbf{r}}$ in Fig. 4.3. In the limit $\Delta t \to 0$ the instantaneous velocity is tangential to the circle.

Magnitude of acceleration in uniform circular motion

For finite Δt we use the similarity of the triangles in Fig. 4.3 to write

$$\frac{|\Delta\vec{\mathbf{v}}|}{v} = \frac{|\Delta\vec{\mathbf{r}}|}{R}. \tag{4.21}$$

The magnitude of the centripetal acceleration is

$$a_c = \lim_{\Delta t\to 0} \frac{|\Delta\vec{\mathbf{v}}|}{\Delta t} = \lim_{\Delta t\to 0} \frac{v}{R}\frac{|\Delta\vec{\mathbf{r}}|}{\Delta t} = v\omega. \tag{4.22}$$

Also, we can derive

$$a_c = \omega^2 R = \frac{v^2}{R} = 4\pi^2 f^2 R = \frac{4\pi^2 R}{T^2}. \tag{4.23}$$

Direction of acceleration in uniform circular motion

Direction of acceleration is decided by the direction of change in velocity $\Delta\vec{\mathbf{v}}$ in Fig. 4.3. In the limit $\Delta t \to 0$ the instantaneous acceleration is radially inward.

Lecture-Example 4.10: (Cloverleaf)

A typical ramp in a cloverleaf interchange design on the interstate has a radius of 50 m. What is the centripetal acceleration of a car exiting an interstate at a speed of 20 m/s ($\sim$ 45 miles/hour). (Answer: 8 m/s^2.) Compare this to the acceleration due to gravity $g = 9.8\,\text{m/s}^2$.

Lecture-Example 4.11: (Trick riding)
In a trick ride a horse is galloping at the speed of 10 m/s, in a circle of radius 6.4 m. What is the centripetal acceleration of the trick rider. (Answer: 16 m/s^2.) Compare this to the acceleration due to gravity $g = 9.8\,\text{m/s}^2$.

Lecture-Example 4.12: (20-G centrifuge, check out this YouTube video.)
The 20-G centrifuge of NASA has a radius of 29 feet (8.8 m). What is the centripetal acceleration at the outer edge of the tube while the centrifuge is rotating at 0.50 rev/sec? (Answer: 9 g.) What is the centripetal acceleration at 0.70 rev/sec? (Answer: 17 g.) Note that such high acceleration causes damage to capillaries, see Table 2.3.

Lecture-Example 4.13: (Gravitropism)
The root tip and shoot tip of a plant have the ability to sense the direction of gravity, very much like smart phones. That is, root tips grow along the direction of gravity, and shoot tips grow against the direction of gravity. (These are associated to statocytes.) Discuss the direction of growth of a plant when placed inside a centrifuge. What if the plant is in zero-gravity? Check out this YouTube video.

Lecture-Example 4.14: (Variation in g)
The acceleration due to gravity is given by, (as we shall derive later in the course,)

$$g = \frac{GM_E}{R_E^2} = 9.82\,\frac{\text{m}}{\text{s}^2}, \tag{4.24}$$

where $M_E = 5.97 \times 10^{24}\,\text{kg}$ and $R_E = 6.37 \times 10^6\,\text{m}$ are the mass and radius of the Earth respectively and $G = 6.67 \times 10^{-11}\,\text{Nm}^2/\text{kg}^2$ is a fundamental constant. This relation does not take into account the rotation of the Earth about its axis and assumes that the Earth is a perfect sphere.

- The centripetal acceleration at a latitude ϕ on the Earth is given by

$$\frac{4\pi^2}{T_E^2} R_E = 0.034\,\cos\phi, \tag{4.25}$$

  where $T_E = 24$ hours is the time period of the Earth's rotation about its axis. It is directed towards the axis of rotation. The component of this acceleration toward the center of the Earth is obtained by multiplying with another factor of $\cos\phi$. The contribution to g from the rotation of the Earth is largest at the equator and zero at the poles.

- The rotation of the Earth has led to its equatorial bulge, turning it into an oblate spheroid. That is, the radius of the Earth at the equator is about 20 km longer than at the poles. This in turn leads to a weaker g at the equator. The fractional change in gravity at a height h above a sphere is approximately, for $h \ll R$, given by $2h/R$. For $h = 42\,\text{km}$ this leads to a contribution of $0.065\,\text{m/s}^2$.

- Contribution to g from rotation of the Earth is positive, and from the equatorial bulge is negative. Together, this leads to the variations in g on the surface of the Earth. Nevertheless, the variations in g are between $9.76\,\mathrm{m/s^2}$ (in the Nevado summit in Peru) and $9.84\,\mathrm{m/s^2}$ (in the Arctic sea), refer this article in Geophysical Research Letters (2013). The measurement of g is relevant for determining the elevation of a geographic location on the Earth. An interesting fact is that even though Mount Everest is the highest elevation above sea level, it is the summit of Chimborazo in Equador that is farthest from the center of the Earth.

Non-uniform circular motion

In terms of unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ we can express the position vector as

$$\vec{\mathbf{r}} = r\,\hat{\mathbf{r}}. \tag{4.26}$$

The velocity is

$$\vec{\mathbf{v}} = \frac{dr}{dt}\,\hat{\mathbf{r}} + r\omega\,\hat{\boldsymbol{\phi}}, \tag{4.27}$$

where we used the chain rule

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\phi}{dt}\frac{d\hat{\mathbf{r}}}{d\phi} = \frac{d\phi}{dt}\hat{\boldsymbol{\phi}}, \tag{4.28}$$

where $\omega = d\phi/dt$. Note that the radial component of velocity is zero for circular motion. The acceleration is

$$\vec{\mathbf{a}} = \left(\frac{d^2r}{dt^2} - \omega^2 r\right)\hat{\mathbf{r}} + \left(r\frac{d^2\phi}{dt^2} + 2\omega\frac{dr}{dt}\right)\hat{\boldsymbol{\phi}}, \tag{4.29}$$

where the term with factor 2 is the Coriolis acceleration that contributes to trade winds. For the case of circular motion we have, using $dr/dt = 0$,

$$\vec{\mathbf{a}} = -\omega^2 r\,\hat{\mathbf{r}} + r\frac{d^2\phi}{dt^2}\,\hat{\boldsymbol{\phi}}. \tag{4.30}$$

Thus, for non-uniform motion in addition to the centripetal acceleration we have a tangential acceleration, which increases or decreases the angular speed of the particle.

Lecture-Example 4.15:
A car is accelerating at $2.0\,\mathrm{m/s^2}$ while driving over a hilltop (that is part of a circle of radius $300\,\mathrm{m}$) at the speed of $25\,\mathrm{m/s}$. Determine the magnitude and direction of the total acceleration of the car when it is passing the hilltop. (Answer: $2.9\,\mathrm{m/s^2}$, pointing 46° below the horizontal.)

4.4 Galilean relativity

Let the relative positions of three particles A, B, and G be related by the relation

$$\vec{\mathbf{r}}_{BG} = \vec{\mathbf{r}}_{BA} + \vec{\mathbf{r}}_{AG}. \tag{4.31}$$

See Fig. 4.4. Taking the derivative with respect to time yields the relation between the respective relative velocities,

$$\vec{\mathbf{v}}_{BG} = \vec{\mathbf{v}}_{BA} + \vec{\mathbf{v}}_{AG}. \tag{4.32}$$

Taking the derivative another time yields the relativity of accelerations as measured by different observers,

$$\vec{\mathbf{a}}_{BG} = \vec{\mathbf{a}}_{BA} + \vec{\mathbf{a}}_{AG}. \tag{4.33}$$

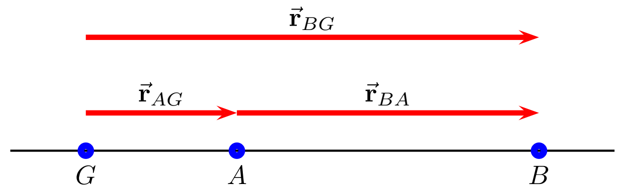

Figure 4.4: Relative positions of three particles at an instant.

The richness and complexity of the seemingly simple idea of relativity is nicely captured in the following 26 minute educational film, titled 'Frames of Reference', released in 1960, starring Profs. Ivey and Hume, and produced by Richard Leacock: https://archive.org/details/frames_of_reference

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Lecture-Example 4.16:
The speedometer of car A measures its speed (with respect to ground) as $\vec{\mathbf{v}}_{AG} = 70\,\hat{\mathbf{i}}$ miles/hour. The speedometer of car B measures its speed (with respect to ground) as $\vec{\mathbf{v}}_{BG} = 60\,\hat{\mathbf{i}}$ miles/hour. Determine the velocity of car B with respect to car A.

- If the initial distance between the cars is 1.0 mile, (with car A trailing car B,) determine the time (in minutes) it will take for car A to overtake car B. (Answer: 6 min.)

Lecture-Example 4.17: (Moving walkway)
Two points inside an airport, separated by a distance of 100.0 m, are connected by a (straight) moving walkway W. The moving walkway has a velocity of $\vec{\mathbf{v}}_{WG} = 3.0\,\hat{\mathbf{i}}$ m/s with respect to the ground G. A person P walks on the moving walkway at a velocity of $\vec{\mathbf{v}}_{PW} = 2.0\,\hat{\mathbf{i}}$ m/s with respect to the walkway. Determine the velocity of the person with respect to the ground $\vec{\mathbf{v}}_{PG}$. (Answer: $5.0\,\hat{\mathbf{i}}$ m/s.)

- Compare the time taken for the person to walk the distance between the two points without using the walkway to that of using the walkway. (Answer: 50 s versus 20 s.)
- Consider a kid P running on the walkway in the opposite direction with velocity $\vec{\mathbf{v}}_{PW} = -4.0\,\hat{\mathbf{i}}$ m/s. Determine the velocity of the kid with respect to the ground $\vec{\mathbf{v}}_{PG}$. (Answer: $-1.0\,\hat{\mathbf{i}}$ m/s.) If the kid starts from one end, determine the time taken for the kid to reach other end of the walkway. (Answer: 100 s.)

Lecture-Example 4.18: (Upstream versus downstream)
A river R is flowing with respect to ground G at a speed of $v_{RG} = 1.5$ m/s. A swimmer S can swim in still water at $v_{SR} = 2.0$ m/s. Determine the time taken by the swimmer to swim a distance of 100.0 m downstream and then swim upstream the same distance, to complete a loop. (Answer: 229 s.)

Lecture-Example 4.19: (Boat crossing a river)
A river R is flowing with respect to ground G with velocity $\vec{\mathbf{v}}_{RG} = 2.0\,\hat{\mathbf{i}}$ m/s. A boat B can move in still water with a speed of $v_{BR} = 6.0$ m/s. The banks of the river are separated by a distance of 200.0 m.

- The boat is moving with respect to river with velocity $\vec{\mathbf{v}}_{BR} = 6.0\,\hat{\mathbf{j}}\,\text{m/s}$. The boat gets drifted. Determine the magnitude and direction of the velocity of the boat with respect to the river. (Answer: 6.3 m/s at an angle 18° clockwise with respect to $\hat{\mathbf{j}}$.) How far down the river will the boat be drifted? (Answer: 67 m.)
- To reach the river right across, at what angle should the boat be directed? (Answer: 20° anticlockwise with respect to $\hat{\mathbf{j}}$.) How much time does it take to reach the shore right across? (Answer: 35 s.)

Lecture-Example 4.20: (Rain)
A train T travels due South at 30 m/s relative to the ground G in a rain R that is blown toward the South by the wind. The path of each raindrop makes an angle of 70° with the vertical, as measured by an observer stationary on the ground. An observer on the train, however, sees the drops fall perfectly vertically. Determine the speed of the raindrops relative to the ground.

Lecture-Example 4.21: (Aeroplane navigation)
An aeroplane A is flying at a speed of 75 m/s with respect to wind W. The wind is flowing at a speed of 20 m/s 30° North of West with respect to ground G. In what direction should the aeroplane head to go due North?

- We have the relation

  $$\vec{\mathbf{v}}_{AG} = \vec{\mathbf{v}}_{AW} + \vec{\mathbf{v}}_{WG}, \tag{4.34}$$

  where we are given

  $$\begin{aligned} \vec{\mathbf{v}}_{AG} &= 0\,\hat{\mathbf{i}} + v_{AG}\,\hat{\mathbf{j}}, && (4.35\text{a}) \\ \vec{\mathbf{v}}_{WG} &= -20\cos 30\,\hat{\mathbf{i}} + 20\sin 30\,\hat{\mathbf{j}}, && (4.35\text{b}) \\ \vec{\mathbf{v}}_{AW} &= 75\cos\alpha\,\hat{\mathbf{i}} + 75\sin\alpha\,\hat{\mathbf{j}}. && (4.35\text{c}) \end{aligned}$$

  This determines the direction to head as $\alpha = 77°$ North of East. The resultant speed of the aeroplane due North is 83 m/s.

Chapter 5

Newton's laws of motion

5.1 Laws of motion

Without precisely defining them, we assume standard notions of force and mass.

Law of inertia: Newton's first law of motion

The concept of inertia is the content of Newton's first law of motion. It states that, a body will maintain constant velocity, unless the net force on the body is non-zero. It is also called the law of inertia. Velocity being a vector, constant here means constant magnitude and constant direction. In other words, a body will move along a straight line, unless acted upon by a force.

An inertial frame is a frame in which the law of inertia holds. A frame that is moving with constant speed with respect to the body is thus an inertial frame, but a frame that is accelerating with respect to the body is not an inertial frame. Einstein extended the law of inertia to non-Euclidean geometries, in which the concept of a straight line is generalized to a geodesic.

Newton's second law of motion

The first law of motion states that a force causes a change in velocity of the body. In the second law the change in velocity is associated to the acceleration of the body. Newton's second law of motion states that for a fixed force the acceleration is inversely proportional to the mass of the body. In this sense mass is often associated to the notion of inertia, because mass resists change in velocity. Newton's second law of motion is expressed using the equation

$$\vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \cdots = m\vec{\mathbf{a}}, \tag{5.1}$$

where m is the mass of the body and $\vec{\mathbf{a}}$ is the acceleration of the body. The left hand side is the vector sum of the individual forces acting on the mass m, which is often conveniently represented by $\vec{\mathbf{F}}_{\text{net}}$.

Newton's third law of motion

A force is exerted by one mass on another mass. Newton's third law states that the other mass exerts an equal and opposite reaction force on the first mass.

5.2 Force of gravity

Near to the surface of Earth a body of mass m experiences a force of gravity given by

$$m\vec{\mathbf{g}}, \tag{5.2}$$

where $|\vec{\mathbf{g}}| = 9.8\,\mathrm{m/s^2}$ and the force $m\vec{\mathbf{g}}$ is directed towards the center of Earth.

Lecture-Example 5.1:
A ball of mass 1.0 kg is dropped above the surface of Earth.

- Determine the magnitude and direction of the acceleration of the ball. (Answer: $9.8\,\mathrm{m/s^2}$ towards the center of Earth.)
- According to Newton's third law the Earth with a mass of $m_E = 5.97 \times 10^{24}\,\mathrm{kg}$ also experiences the same force in the opposite direction. Determine the magnitude and direction of the acceleration of the Earth as a result. (Answer: $1.6 \times 10^{-24}\,\mathrm{m/s^2}$ towards the ball.)

5.3 Normal force

Due to the gravitational force acting on a body its tendency is to accelerate towards the center of Earth. This tendency is resisted when the body comes in contact with the surface of another body. The component of the force normal (perpendicular) to the plane of the surface is called the normal force, and is often represented by $\vec{\mathbf{N}}$. Typical weighing scale, using a spring, measures the normal force, which is then divided by $9.8\,\mathrm{m/s^2}$ to report the mass.

Lecture-Example 5.2: (Normal force)
A body of mass $m = 10.0\,\mathrm{kg}$ rests on a weighing scale on a horizontal table.

- Determine the magnitude of the normal force acting on the mass. (Answer: 98 N.)
- Determine the magnitude of the normal force acting on the mass while you push on it vertically downwards with a force of 20 N. (Answer: 120 N.) Determine the reading on the scale. (Answer: 12 kg.)
- Determine the magnitude of the normal force acting on the mass while you pull on it vertically upwards with a force of 20 N. (Answer: 78 N.) Determine the reading on the scale. (Answer: 8.0 kg.)
- Determine the magnitude of the normal force acting on the mass while you pull on it vertically upwards with a force of 98 N. (Answer: 0 N.) Determine the reading on the scale. (Answer: 0 kg.)
- Determine the magnitude of the normal force acting on the mass while you pull on it vertically upwards with a force of 150 N. (Answer: 0 N.) Describe what happens. (Answer: The mass will accelerate upwards at $5.3\,\mathrm{m/s^2}$.)

Lecture-Example 5.3: (Elevator)
Your mass is 75 kg. How much will you weigh on a bathroom scale (designed to measure the normal force in Newtons) inside an elevator that is

- at rest? (Answer: 740 N.)
- moving upward at constant speed? (Answer: 740 N.)
- moving downward at constant speed? (Answer: 740 N.)
- slowing down at $2.0\,\mathrm{m/s^2}$ while moving upward? (Answer: 590 N.)

- speeding up at 2.0 m/s^2 while moving upward? (Answer: 890 N.)
- slowing down at 2.0 m/s^2 while moving downward? (Answer: 890 N.)
- speeding up at 2.0 m/s^2 while moving downward? (Answer: 590 N.)

Lecture-Example 5.4: (Weight on anincline)
Your mass is 75 kg (or 735 Newtons). How much will you weigh on a weighing scale (designed to measure the normal force in Newtons) while standing on an incline making an angle of 30° with the horizontal.

Lecture-Example 5.5: (Frictionless incline)
A mass m is on a frictionless incline that makes an angle θ with the horizontal. Let $m = 25.0$ kg and $\theta = 30.0°$.

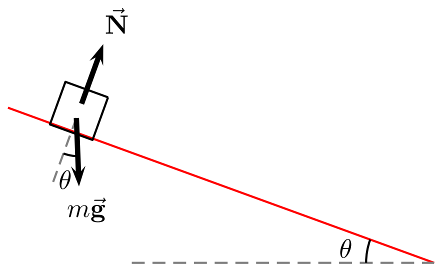

Figure 5.1: Lecture-Example 5.5

- Using Newton's law determine the equations of motion to be

$$mg \sin\theta = ma, \tag{5.3a}$$
$$N - mg\cos\theta = 0. \tag{5.3b}$$

- Determine the normal force. (Answer: $N = 212$ N.)
- Determine the acceleration of the mass. (Answer: $a = 4.9$ m/s^2.) How does the acceleration of the mass change if the mass is heavier or lighter?
- Starting from rest how long does the mass take to travel a distance of 3.00 m along the incline? (Answer: 1.1 s.)
- The optical illusion, The Demon Hill, by the artist Julian Hoeber, presumably motivated by naturally occuring 'Mystery Spots', are based on this idea. Check out this video:

https://www.youtube.com/watch?v=1BMSYXK4-AI (5:16 minutes)

Lecture-Example 5.6:
A mass m is pulled on a frictionless surface by a force $\vec{\mathbf{F}}_{\rm pull}$ that makes an angle θ with the horizontal. Let $m = 25.0$ kg, $F_{\rm pull} = 80.0$ N, and $\theta = 30.0°$.

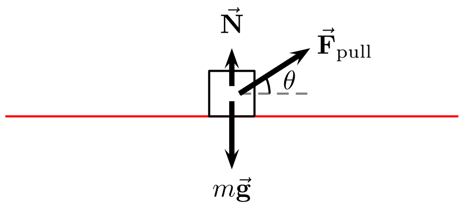

Figure 5.2: Lecture-Example 5.6

- Using Newton's law determine the equations of motion to be

$$F_{\rm pull}\cos\theta = ma_x, \tag{5.4a}$$
$$N + F_{\rm pull}\sin\theta - mg = 0. \tag{5.4b}$$

- Determine the normal force. (Answer: $N = 205\,\mathrm{N}$.)
- Determine the acceleration of the mass. (Answer: $a_x = 2.77\,\mathrm{m/s^2}$.) Starting from rest how far does the mass move in one second?
- Discuss what happens if θ above the horizontal is increased.
- Discuss what happens if θ is below the horizontal.

Lecture-Example 5.7: (Three masses)
Three masses $m_1 = 10.0\,\mathrm{kg}$, $m_2 = 20.0\,\mathrm{kg}$, and $m_3 = 30.0\,\mathrm{kg}$, are stacked together on a frictionless plane. A force $\vec{\mathbf{F}}$ is exerted on m_1.

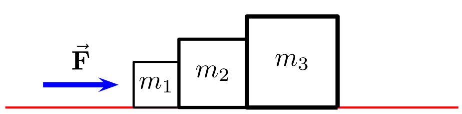

Figure 5.3: Lecture-Example 5.7

- Using Newton's law determine the equations of motion to be

$$F - C_{12} = m_1 a, \qquad N_1 = m_1 g, \tag{5.5a}$$
$$C_{21} - C_{23} = m_2 a, \qquad N_2 = m_2 g, \qquad C_{21} = C_{12}, \tag{5.5b}$$
$$C_{32} = m_3 a, \qquad N_3 = m_3 g, \qquad C_{32} = C_{23}. \tag{5.5c}$$

  Here C_{ij} are contact forces acting on i by j. Thus, determine the acceleration and contact forces to be

$$a = \frac{F}{(m_1 + m_2 + m_3)}, \tag{5.6a}$$
$$C_{12} = C_{21} = \frac{(m_2 + m_3)F}{(m_1 + m_2 + m_3)} = \frac{5}{6}F, \tag{5.6b}$$
$$C_{23} = C_{32} = \frac{m_3 F}{(m_1 + m_2 + m_3)} = \frac{1}{2}F. \tag{5.6c}$$

- Show that if the force $\vec{\mathbf{F}}$ were exerted on mass m_3 instead we have

$$C_{12} = C_{21} = \frac{m_1 F}{(m_1 + m_2 + m_3)} = \frac{1}{6} F, \tag{5.7a}$$

$$C_{23} = C_{32} = \frac{(m_1 + m_2) F}{(m_1 + m_2 + m_3)} = \frac{1}{2} F. \tag{5.7b}$$

while the acceleration remains the same. Discuss the difference in the stresses on the surfaces of contact in the two cases.

5.4 Force due to tension in strings

Ropes and strings exert forces due to tension in them. In most of discussions we will assume the mass of the rope to be negligible in comparison to the masses of the moving bodies. That is we pretend the strings to be of zero mass.

Lecture-Example 5.8: (Double mass)
Two masses m_1 and m_2 are hanging from two ropes as described in Figure 5.4.

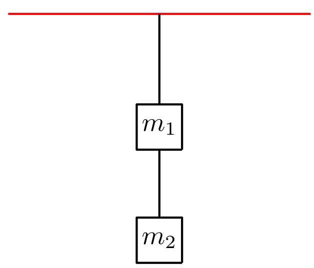

Figure 5.4: Lecture-Example 5.8

- Using Newton's law determine the equations of motion to be

$$T_1 - T_2 = m_1 g, \tag{5.8a}$$

$$T_2 = m_2 g. \tag{5.8b}$$

Thus, show that

$$T_1 = (m_1 + m_2) g, \tag{5.9a}$$

$$T_2 = m_2 g. \tag{5.9b}$$

- Which rope has the larger tension in it? If the two ropes are identical, which rope will break first if the mass m_2 is gradually increased?

Lecture-Example 5.9: (Atwood's machine)
The Atwood machine consists of two masses m_1 and m_2 connected by a massless (inextensible) string passing over a massless pulley. See Figure 5.5.

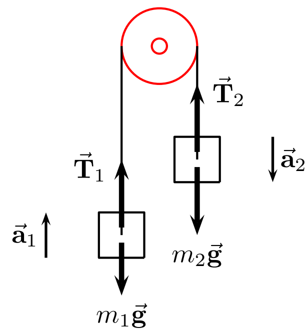

Figure 5.5: Lecture-Example 5.9

- Massless pulley implies that $|\vec{\mathbf{T}}_1| = |\vec{\mathbf{T}}_2| = T$. And, inextensible string implies that $|\vec{\mathbf{a}}_1| = |\vec{\mathbf{a}}_2| = a$.
- Using Newton's law determine the equations of motion to be

$$m_2 g - T = m_2 a, \tag{5.10a}$$
$$T - m_1 g = m_1 a. \tag{5.10b}$$

Thus, show that

$$a = \left(\frac{m_2 - m_1}{m_2 + m_1}\right) g, \tag{5.11a}$$
$$T = \frac{2m_1 m_2 g}{(m_1 + m_2)}. \tag{5.11b}$$

- Starting from rest how far do the masses move in a certain amount of time?
- Determine the acceleration for $m_2 \gg m_1$ and describe the motion? Determine the acceleration for $m_2 \ll m_1$ and describe the motion? Plot a as a function of m_2 for fixed m_1.

Lecture-Example 5.10:
A mass is held above ground using two ropes as described in Figure 5.6. Let $m = 20.0\,\text{kg}$, $\theta_1 = 30.0°$, and $\theta_2 = 60.0°$.

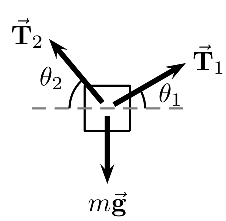

Figure 5.6: Lecture-Example 5.10

- Using Newton's law determine the equations of motion to be

$$T_1 \sin\theta_1 + T_2 \sin\theta_2 = mg, \tag{5.12a}$$
$$T_1 \cos\theta_1 - T_2 \cos\theta_2 = 0. \tag{5.12b}$$

Then, solve these equations to find

$$T_1 = \frac{mg\cos\theta_2}{\sin(\theta_1+\theta_2)}, \tag{5.13a}$$
$$T_2 = \frac{mg\cos\theta_1}{\sin(\theta_1+\theta_2)}. \tag{5.13b}$$

Which rope has the larger tension in it? If the two ropes are identical, which rope will break first if the mass is gradually increased?

- For the special case of $\theta_1 + \theta_2 = \pi/2$ verify that $mg = \sqrt{T_1^2 + T_2^2}$.

Lecture-Example 5.11:
A mass $m_2 = 2.0\,\text{kg}$ is connected to another mass $m_1 = 1.0\,\text{kg}$ by a massless (inextensible) string passing over a massless pulley, as described in Figure 5.7. Assume frictionless surfaces.

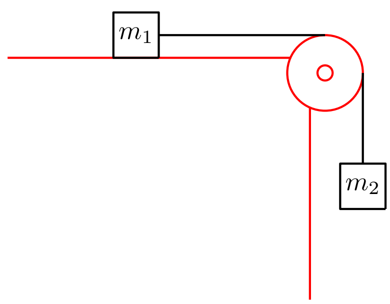

Figure 5.7: Lecture-Example 5.11

- Using Newton's law determine the equations of motion to be

$$m_2 g - T = m_2 a, \tag{5.14a}$$
$$T = m_1 a, \tag{5.14b}$$
$$N_1 = m_1 g. \tag{5.14c}$$

Thus, show that

$$a = \frac{m_2 g}{m_2 + m_1}, \tag{5.15a}$$
$$T = \frac{m_1 m_2 g}{m_1 + m_2}, \tag{5.15b}$$
$$N_1 = m_1 g. \tag{5.15c}$$

- Starting from rest how far do the masses move in a certain amount of time?

- Determine the acceleration for $m_2 \gg m_1$ and describe the motion? Determine the acceleration for $m_2 \ll m_1$ and describe the motion? Plot a as a function of m_2 for fixed m_1.

Lecture-Example 5.12: (Double incline)
A mass $m_2 = 2.0\,\text{kg}$ is connected to another mass $m_1 = 1.0\,\text{kg}$ by a massless (inextensible) string passing over a massless pulley, as described in Figure 5.8. Surfaces are frictionless.

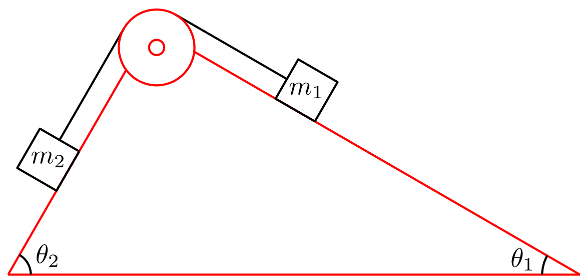

Figure 5.8: Lecture-Example 5.12

- Using Newton's law determine the equations of motion to be

$$m_1 g \sin\theta_1 - T = m_1 a, \qquad N_1 = m_1 g \cos\theta_1, \tag{5.16a}$$
$$T - m_2 g \sin\theta_2 = m_2 a, \qquad N_2 = m_2 g \cos\theta_2. \tag{5.16b}$$

Thus, show that

$$a = \frac{(m_1 \sin\theta_1 - m_2 \sin\theta_2)}{(m_1 + m_2)} g, \tag{5.17a}$$
$$T = \frac{m_1 m_2 (\sin\theta_1 + \sin\theta_2)}{(m_1 + m_2)} g. \tag{5.17b}$$

- Starting from rest how far do the masses move in a certain amount of time?
- Show that for $\theta_1 = \theta_2 = \pi/2$ the results for Atwood machine are reproduced.
- Show that the masses do not accelerate when $m_1 \sin\theta_1 = m_2 \sin\theta_2$. They accelerate to the right when $m_1 \sin\theta_1 > m_2 \sin\theta_2$, and they accelerate to the left when $m_1 \sin\theta_1 < m_2 \sin\theta_2$.

5.5 Force of friction

While two solid surfaces are in contact, the force of friction is the force that resists the tendency of the surfaces to move relative to each other in the lateral direction (parallel to the surface). It acts in the direction opposite to the direction of tendency of motion.

We shall use an empirical model, by Coulomb, to model the force of friction. The Coulomb model assumes that the force of friction is independent of the apparent contact area between two surfaces. Instead it depends on the effective contact area between the two surfaces at the microscopic level. The effective contact area is typically less than the apparent contact area, but it could be more too. The Coulomb model assumes that the

Surface 1	Surface 2	μ_s	μ_k
Concrete	Rubber	1.0(dry), 0.3(wet)	0.6
Metal	Wood	0.4	0.3
Metal	Ice	0.02	0.01

Table 5.1: Approximate coefficients of friction between surfaces.

effective contact area is proportional to the normal force between the two surfaces. In particular, the Coulomb model states that

$$F_f = \begin{cases} F, & F < \mu_s N, \quad (a = 0, \text{ static case}), \\ \mu_s N, & F = \mu_s N, \quad (a = 0, \text{ static case}), \\ \mu_k N, & F > \mu_s N, \quad (a > 0, \text{ kinetic case}), \end{cases} \tag{5.18}$$

where F is the sum of all forces excluding the force of friction.

Lecture-Example 5.13:
A $m = 20.0\,\text{kg}$ ($mg = 196\,\text{N}$) block is at rest on a horizontal floor. The coefficient of static friction between the floor and the block is 0.50, and the coefficient of kinetic friction between the floor and the block is 0.40.

- What is the normal force N exerted on the block by the floor? (Answer: 196 N.)
- Calculate the maximum static frictional force, $F_{f,\text{max}} = \mu_s N$, possible between the block and floor. (Answer: 98 N.)
- Calculate the kinetic frictional force, $F_f = \mu_k N$, between the block and floor if the block moves on the floor. (Answer: 78 N.)
- While the block is initially at rest you exert a horizontal force of 85 N on the block. Will the block move? (Answer: No.)
- While the block is initially at rest you exert a horizontal force of 105 N on the block. Will the block move? If yes, what will be it's acceleration? (Answer: Yes, $a = 1.35\,\text{m/s}^2$.)

Lecture-Example 5.14:
A trunk with a weight of 196 N rests on the floor. The coefficient of static friction between the trunk and the floor is 0.50, and the coefficient of kinetic friction is 0.40.

- What is the magnitude of the minimum horizontal force with which a person must push on the trunk to start it moving? (Answer: 98 N.)
- Once the trunk is moving, what magnitude of horizontal force must the person apply to keep it moving with constant velocity? (Answer: 78.4 N.)
- If the person continued to push with the force used to start the motion, what would be the magnitude of the trunk's acceleration? (Answer: $0.98\,\text{m/s}^2$.)

Lecture-Example 5.15:
A car is traveling at 70.0 miles/hour (= 31.3 m/s) on a horizontal highway.

- What is the stopping distance when the surface is dry and the coefficient of kinetic friction μ_s between road and tires is 0.60? (Answer: 83 m.)
- If the coefficient of kinetic friction between road and tires on a rainy day is 0.40, what is the minimum distance in which the car will stop? (Answer: 125 m.)

Lecture-Example 5.16:
A mass $m = 20.0\,\text{kg}$ is on an incline with coefficient of static friction $\mu_s = 0.80$ and coefficient of kinetic friction $\mu_k = 0.50$.

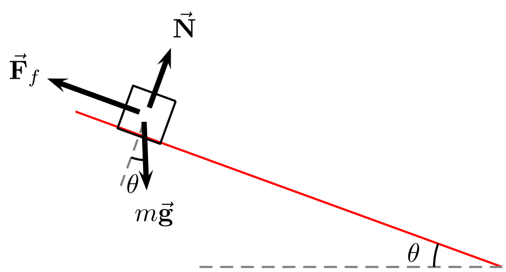

Figure 5.9: Lecture-Example 5.16

- Using Newton's law determine the equations of motion to be, choosing the x axis to be parallel to the incline,

$$mg\sin\theta - F_f = ma_x, \tag{5.19a}$$
$$N - mg\cos\theta = 0. \tag{5.19b}$$

- Let $\theta = 30.0°$. Determine the normal force. (Answer: 170 N.) Determine the maximum static frictional force, $F_{f,\text{max}} = \mu_s N$, possible between the mass and the incline. (Answer: $F_{f,\text{max}} = 136\,\text{N}$.) Find the net force in the lateral direction other than friction. (Answer: $mg\sin\theta = 98\,\text{N}$.) Determine the force of friction on the mass. (Answer: 98 N.) Will the mass move? (Answer: No.)
- Let $\theta = 45.0°$. Determine the normal force. (Answer: 139 N.) Determine the maximum static frictional force, $F_{f,\text{max}} = \mu_s N$, possible between the mass and the incline. (Answer: $F_{f,\text{max}} = 111\,\text{N}$.) Find the net force in the lateral direction other than friction. (Answer: $mg\sin\theta = 139\,\text{N}$.) Determine the force of friction on the mass. (Answer: $F_f = \mu_k N = 70\,\text{N}$.) Will the mass move? (Answer: Yes.) Determine the acceleration of the resultant motion. (Answer: $3.5\,\text{m/s}^2$.)
- Critical angle: As the angle of the incline is increased, there is a critical angle when the mass begins to move. For this case the force of friction is equal to the maximum static frictional force, $F_f = \mu_s N$, and the mass is at the verge of moving, $a_x = 0$. Show that the critical angle is given by

$$\theta_c = \tan^{-1}\mu_s, \tag{5.20}$$

  which is independent of the mass m. (Answer: $\theta_c = 38.7°$.)
- Concept question: Consider the case of a bucket resting on the inclined roof of a house. It starts to rain and the bucket gradually fills with water. Assuming a constant coefficient of static friction between the roof and bucket, no wind, and no tipping, when will the bucket start sliding?
- Concept question: A block is projected up a frictionless inclined plane with initial speed v_0. The angle of incline is $\theta = 30.0°$. Will the block slide back down?

Lecture-Example 5.17:
A mass m is held to a vertical wall by pushing on it by a force $\vec{\mathbf{F}}$ exerted an angle θ with respect to the vertical.

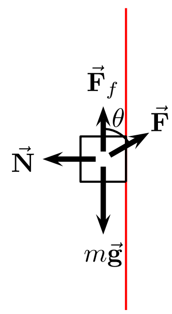

Figure 5.10: Lecture-Example 5.17

- Using Newton's law determine the equations of motion to be,

$$F\sin\theta - N = 0, \tag{5.21a}$$
$$F_f - F\cos\theta - mg = 0. \tag{5.21b}$$

Show that the inequality to be satisfied, for the mass to be held up, is given by

$$mg \le F(\cos\theta + \mu_s \sin\theta). \tag{5.22}$$

Chapter 6

Newton's laws of motion (contd.)

6.1 Uniform circular motion

A particle uniformly moving along a circular path is accelerating radially inward, given by

$$\vec{\mathbf{a}} = -\frac{v^2}{R}\,\hat{\mathbf{r}}, \tag{6.1}$$

where $\hat{\mathbf{r}}$ is a unit vector pointing radially outward, R is the radius of the circle, and v is the magnitude of the uniform velocity. Newton's law then implies that the sum of the total force acting on the system necessarily has to point radially inward.

Lecture-Example 6.1:
A stuntman drives a car over the top of a hill, the cross section of which can be approximated by a circle of radius $R = 250\,\text{m}$. What is the greatest speed at which he can drive without the car leaving the road at the top of the hill?

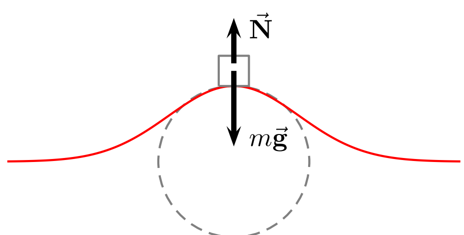

Figure 6.1: Lecture-Example 6.1

Lecture-Example 6.2:
A turntable is rotating with a constant angular speed of 6.5 rad/s. You place a penny on the turntable.

- List the forces acting on the penny.
- Which force contributes to the centripetal acceleration of the penny?
- What is the farthest distance away from the axis of rotation of the turntable that you can place a penny such that the penny does not slide away? The coefficient of static friction between the penny and the turntable is 0.5.

Lecture-Example 6.3: (Motorcycle stunt)
In the Globe of Death stunt motorcycle stunt riders ride motorcycles inside a mesh globe. In particular, they can loop vertically. Consider a motorcycle going around a vertical circle of radius R, inside the globe, with uniform velocity. Determine the normal force and the force of friction acting on the motorcycle as a function of angle θ described in Figure 6.2.

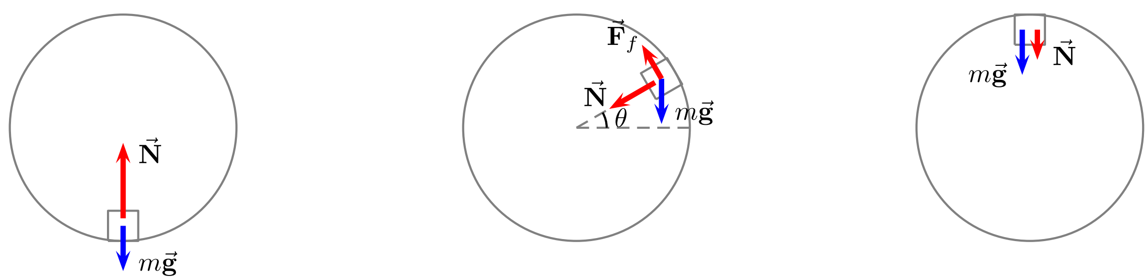

Figure 6.2: Forces acting on a mass while moving in a vertical circle inside a globe.

- Using Newton's Laws we have the equations of motion, along the radial and tangential direction to the circle, given by

$$N = \frac{mv^2}{R} - mg\cos\theta, \tag{6.2a}$$

$$F_f = mg\sin\theta. \tag{6.2b}$$

- Investigate the magnitude and direction of the normal and force of friction as a function of angle θ. In particular, determine these forces for $\theta = 0, -90°, 90°$. Verify that, while at $\theta = 90°$, the motorcycle can not stay there without falling off unless the the centripetal acceleration is sufficiently high, that is, $mv^2/R \geq mg$.

6.2 Banking of roads

Motorized cars are all around us, and we constantly encounter banked roads while driving on highways bending along a curve. A banked road is a road that is appropriately inclined, around a turn, to reduce the chances of vehicles skidding while maneuvering the turn. Banked roads are more striking in the case of racetracks on which the race cars move many times faster than typical cars on a highway. Nevertheless, this ubiquitous presence of banked roads around us does not lessen the appreciation for this striking application of Newton's laws.

Unbanked frictionless surface

A car can not drive in a circle on an unbanked frictionless surface, because there is no horizontal force available to contribute to the (centripetal) acceleration due to circular motion.

Unbanked surface with friction

Consider a car moving with uniform speed along a circular path of radius R on a flat surface with coefficient of static friction μ_s. Using Newton's laws we have the equations of motion

$$F_f = \frac{mv^2}{R}, \tag{6.3a}$$

$$N = mg, \tag{6.3b}$$

where $F_f \leq \mu_s N$. The maximum speed the car can achieve without sliding is given by

$$v_{\rm max}^2 = gR\tan\theta_s, \tag{6.4}$$

where we used the definition of friction angle $\mu_s = \tan\theta_s$.

Banked frictionless surface

Let the surface make an angle θ with respect to the horizontal. Even though there is no friction force due to the geometry of the banking the normal force is able to provide the necessary centripetal acceleration. Using Newton's laws we have the equations of motion

$$N\sin\theta = \frac{mv^2}{R}, \tag{6.5a}$$

$$N\cos\theta = mg. \tag{6.5b}$$

The speed of the car is given by

$$v^2 = gR\tan\theta. \tag{6.6}$$

Thus, if the car speeds up it automatically gets farther away and vice versa.

Banked surface with friction

Let us now consider the case of a banked surface with friction. In this case both the normal force and the force of friction are available to contribute to the centripetal acceleration. There now exists a particular speed v_0 that satisfies

$$v_0^2 = gR\tan\theta, \tag{6.7}$$

for which case the normal force alone completely provides the necessary centripetal force and balances the force of gravity, see Figure 6.3. Thus, in this case, the frictional force is completely absent, as illustrated in Figure 6.3. The physical nature of the problem, in the sense governed by the direction of friction, switches sign at speed v_0.

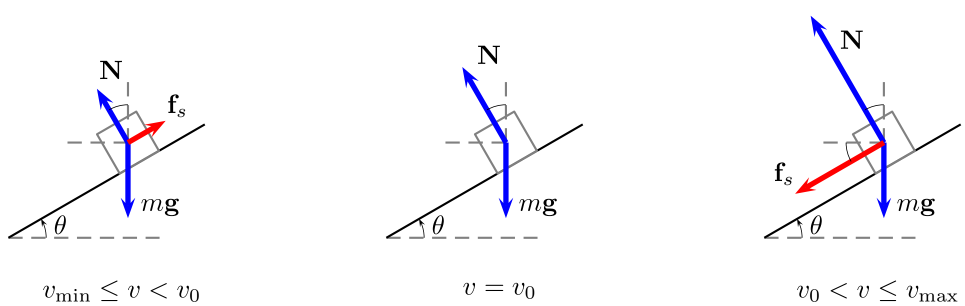

Figure 6.3: Forces acting on a car moving on a banked road. The car is moving into the page. The direction of friction is inward for $v_0 < v \leq v_{\rm max}$, outward for $v_{\rm min} < v < v_0$, and zero for $v = v_0$.

Let us begin by investigating what happens when the car deviates from this speed v_0? If the speed of the car is different from v_0, the normal force alone cannot provide the necessary centripetal acceleration without sliding. Thus, as a response, the frictional force gets switched on. The frictional force responds to act (inwards) when the car moves faster than v_0; this provides the additional force necessary to balance the centripetal force, see Figure 6.3. Similarly, the frictional force acts in the negative direction (outwards) when the car moves slower than v_0, see Figure 6.3. Let the frictional force be represented by $\vec{\mathbf{F}}_f$. Thus, for the case when the frictional force is acting inward, we have the equations of motion for the car given by,

$$N \sin\theta + F_f \cos\theta = \frac{mv^2}{R}, \tag{6.8a}$$

$$N \cos\theta - F_f \sin\theta = mg. \tag{6.8b}$$

The equations of motion for the car when the frictional force is acting outward are given by Eqs. (6.8) by changing the sign of F_f. Can the frictional force together with the normal force balance the centripetal force for all speeds? No. There exists an upper threshold to speed $v_{\rm max}$ beyond which the frictional force fails to balance the centripetal force, and it causes the car to skid outward. Similarly, there exists a lower threshold to speed $v_{\rm min}$ below which the car skids inward. To this end it is convenient to define

$$F_f \le \mu_s N, \qquad \mu_s = \tan\theta_s, \tag{6.9}$$

where μ_s is the coefficient of static friction, and θ_s is a suitable reparametrization of the coefficient of static friction. The upper threshold for the speed is obtained by using the equality of Eq. (6.9) in Eq. (6.8) to yield

$$v_{\rm max}^2 = rg\tan(\theta + \theta_s), \tag{6.10}$$

where we used the definition in Eq. (6.9) and the trigonometric identity for the tangent of the sum of two angles. Similarly, the lower threshold for the speed below which the car slides inward is given by

$$v_{\rm min}^2 = rg\tan(\theta - \theta_s). \tag{6.11}$$

In summary, at any given point on the surface of the cone, to avoid skidding inward or outward in the radial direction, the car has to move within speed limits described by

$$v_{\rm min} \le v \le v_{\rm max}. \tag{6.12}$$

6.3 Drag forces

Friction forces that are proportional to velocity are called drag forces. Let us consider the case when the friction force is linearly proportional to velocity,

$$\vec{\mathbf{F}}_f = -b\vec{\mathbf{v}}. \tag{6.13}$$

For a mass m falling under gravity we have the equation of motion

$$m\frac{dv}{dt} = mg - bv. \tag{6.14}$$

As the mass falls it gains speed and the frictional force eventually balances the force of gravity, and from this point on it does not accelerate. Thus, the terminal velocity is defined by requiring $dv/dt = 0$, that is

$$v_T = \frac{mg}{b}. \tag{6.15}$$

The equation of motion can be solved for the initial condition of the particle starting from rest, $v(0) = 0$, which leads to the solution

$$v(t) = v_T\Big(1 - e^{-\frac{t}{\tau}}\Big), \tag{6.16}$$

where $\tau = v_T/g$ sets the scale for time. We make the observation that the particle never reaches the terminal speed, it approaches it at infinite time.

Lecture-Example 6.4: (Time constant $F_f \propto v$)
Show that the velocity of the particle at time $t = \tau = v_T/g$, during the fall is

$$v(\tau) = v_T\left(1 - \frac{1}{e}\right) \sim 0.632\, v_T. \tag{6.17}$$

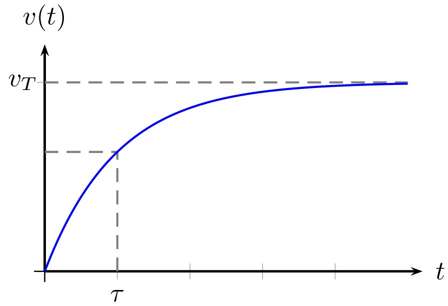

Figure 6.4: Terminal velocity for $F_f \propto v$.

- Evaluate the time constant τ for the case $v_T = 1.0\,\text{cm/s}$. (Answer: $\tau = 1.0\,\text{ms}$.)

Let us consider the case when the friction force is quadratically proportional to velocity,

$$F_f = \frac{1}{2} D\rho A v^2, \tag{6.18}$$

where A is the area of crosssection, ρ is the density of the medium, and D is the dimensionless drag coefficient. For a mass m falling under gravity we have the equation of motion

$$m\frac{dv}{dt} = mg - F_f. \tag{6.19}$$

The terminal velocity, when $dv/dt = 0$, now is given by

$$v_T = \sqrt{\frac{2mg}{D\rho A}}. \tag{6.20}$$

The equation of motion can be solved for the initial condition of the particle starting from rest, $v(0) = 0$, which leads to the solution

$$v(t) = v_T \frac{\left(1 - e^{-\frac{2t}{\tau}}\right)}{\left(1 + e^{-\frac{2t}{\tau}}\right)}, \tag{6.21}$$

where $\tau = v_T/g$ again sets the scale for time.

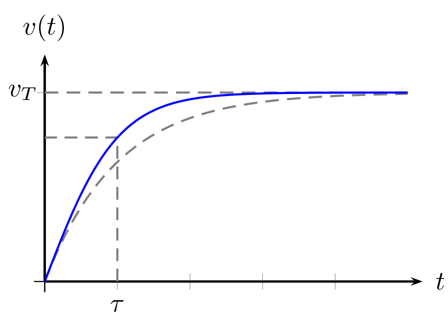

Figure 6.5: Terminal velocity for $F_f \propto v^2$.

Lecture-Example 6.5: (Time constant for $F_f \propto v^2$)

Show that the velocity of the particle at time $t = \tau = v_T/g$, during the fall is

$$v(\tau) = v_T \frac{\left(1 - \frac{1}{e^2}\right)}{\left(1 - \frac{1}{e^2}\right)} \sim 0.762\, v_T. \tag{6.22}$$

Chapter 7

Work and Energy

7.1 Scalar product

Scalar product of two vectors

$$\vec{\mathbf{A}} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}, \tag{7.1a}$$
$$\vec{\mathbf{B}} = B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}} + B_z\,\hat{\mathbf{k}}, \tag{7.1b}$$

is given by

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos\theta = A_xB_x + A_yB_y + A_zB_z, \tag{7.2}$$

where θ is the angle between the two vectors. The scalar product is a measure of the component of one vector along another vector.

7.2 Work-energy theorem

Starting from Newton's law

$$\vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \ldots = m\vec{\mathbf{a}}, \tag{7.3}$$

and integrating on both sides along the path of motion, we derive the work-energy theorem

$$W_1 + W_2 + \ldots = \Delta K, \tag{7.4}$$

where W_i is the work done by the force $\vec{\mathbf{F}}_i$, $(i = 1, 2, \ldots,)$ and ΔK is the change in kinetic energy.

Work done by a force

Work done by a force $\vec{\mathbf{F}}$ on mass m while displacing it from an initial point $\vec{\mathbf{r}}_i$ to a final $\vec{\mathbf{r}}_f$, along a path P, is given by

$$W = \int_{\vec{\mathbf{r}}_i, P}^{\vec{\mathbf{r}}_f} d\vec{\mathbf{r}} \cdot \vec{\mathbf{F}}. \tag{7.5}$$

Work done is measured in the units of energy, Joule = Newton · meter.

Kinetic energy

The energy associated with the state of motion, the kinetic energy, is

$$K = \frac{1}{2}mv^2, \tag{7.6}$$

where v is the magnitude of the velocity of mass m.

Lecture-Example 7.1: (Area under the force-position graph.)
Consider the motion of a mass m under the action of a force

$$F = -kx, \tag{7.7}$$

where k is a constant. Show that the work done by the force is equal to the area under the force-position graph.

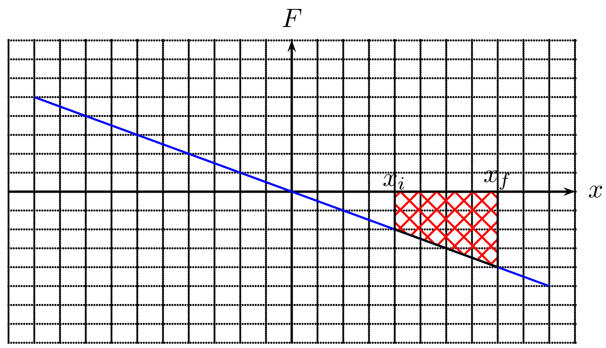

Figure 7.1: A Force-position graph.

- The work done by the force is

$$W = \int_{x_i}^{x_f} (-kx)\,dx = -\frac{1}{2}k(x_f^2 - x_i^2). \tag{7.8}$$

- Show that the area under the force-position graph is the sum of the area of a triangle and a rectangle,

$$W = -\frac{1}{2}k(x_f - x_i)^2 - kx_i(x_f - x_i). \tag{7.9}$$

Lecture-Example 7.2:
Consider a mass $m = 25\,\text{kg}$ being pulled by a force $F_{\text{pull}} = 80.0\,\text{N}$, exerted horizontally, such that the mass moves, on a horizontal surface with coefficient of kinetic friction $\mu_k = 0.30$. Assume that the mass starts from rest. We would like to determine the final velocity v_f after the mass has moved a horizontal distance $d = 10.0\,\text{m}$.

- We identify four forces acting on the mass and write Newton's law for the configuration as

$$m\vec{\mathbf{g}} + \vec{\mathbf{N}} + \vec{\mathbf{F}}_{\text{pull}} + \vec{\mathbf{F}}_f = m\vec{\mathbf{a}}. \tag{7.10}$$

- Work done by the individual force are

$$W_{\text{pull}} = F_{\text{pull}} d \cos 0 = F_{\text{pull}} d = 800\,\text{J}, \tag{7.11a}$$
$$W_g = mgd \cos 90 = 0\,\text{J}, \tag{7.11b}$$
$$W_N = Nd \cos 90 = 0\,\text{J}, \tag{7.11c}$$
$$W_f = F_f d \cos 180 = -F_f d = -\mu_k N d = -\mu_k mgd = -735\,\text{J}, \tag{7.11d}$$

where we used $F_f = \mu_k N$, and then used Newton's law in the vertical y-direction to learn that $N = mg$.

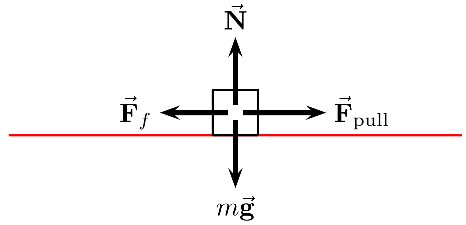

Figure 7.2: Lecture-Example 7.2

- The total work done by the sum of all the forces is

$$W_{pull} + W_g + W_N + W_f = F_{pull} d - \mu_k m g d = 65\,\text{J}. \tag{7.12}$$

- Using the work-energy theorem and using $v_i = 0$ we have

$$W_{pull} + W_g + W_N + W_f = \frac{1}{2} m v_f^2. \tag{7.13}$$

  Using Eq. (7.12) we then have

$$F_{pull} d - \mu_k m g d = \frac{1}{2} m v_f^2. \tag{7.14}$$

  Substituting numbers we can determine $v_f = 2.28\,\text{m/s}$.

Lecture-Example 7.3:
Consider a mass $m = 25\,\text{kg}$ being pulled by a force $F_{pull} = 80.0\,\text{N}$, exerted along a line making angle $\theta = 30.0°$ above the horizontal, such that the mass moves, on a horizontal surface with coefficient of kinetic friction $\mu_k = 0.30$. Assume that the mass starts from rest. Determine the final velocity v_f after the mass has moved a horizontal distance $d = 10.0\,\text{m}$.

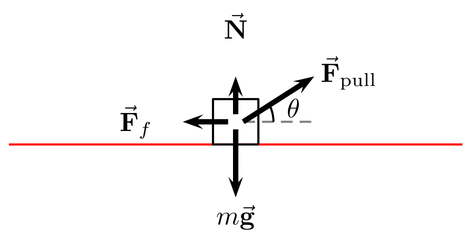

Figure 7.3: Lecture-Example 7.3

- The work done by the individual forces are

$$\begin{aligned} W_{pull} &= F_{pull} d \cos\theta = 693\,\text{J}. &(7.15a)\\ W_g &= m g d \cos 90 = 0\,\text{J}. &(7.15b)\\ W_N &= N d \cos 90 = 0\,\text{J}. &(7.15c)\\ W_f &= F_f d \cos 180 = -\mu_k N d = -\mu_k (m g - F_{pull}\sin\theta) d = -615\,\text{J}. &(7.15d) \end{aligned}$$

  We used $N = mg - \mu_k F_{pull} \sin\theta$, a deduction from the y-component of Newton's law.

- Using the work energy theorem we obtain

$$K_f = F_{\text{pull}} d(\cos\theta + \mu_k \sin\theta) - \mu_k mgd, \tag{7.16}$$

which leads to $v_f = 2.50\,\text{m/s}$.

- Discussion: Observe that this velocity is greater than the velocity calculated for the case $\theta = 0°$, after Eq. (7.14). When exerted at an angle the force contribution is less in the direction of motion. But, the normal force decreases in this case and leads to reduction in the friction too. This suggests that there is an optimum angle θ for which the final velocity is maximum. Presuming $K_f > 0$ and $N > 0$, determine the angle θ_{max} when the final velocity is a maximum. This is determined by the condition

$$\frac{\partial K_f}{\partial \theta} = 0, \tag{7.17}$$

which is satisfied when

$$\tan\theta_{\text{max}} = \mu_k, \tag{7.18}$$

corresponding to $\theta_{\text{max}} = 16.7°$. Show that the maximum velocity and thus the maximum kinetic energy is then given by

$$K_f^{\text{max}} = F_{\text{pull}} d\sqrt{1 + \mu_k^2} - \mu_k mgd. \tag{7.19}$$

Lecture-Example 7.4:
A mass $m = 25\,\text{kg}$ slides down an inclined plane with angle $\theta = 30.0°$. Assume coefficient of kinetic friction $\mu_k = 0.30$. Assume that the mass starts from rest. Determine the final velocity v_f after the mass has moved a distance $d = 10.0\,\text{m}$ along the incline.

- Determine the work done by the three individual forces.

- Using the work-energy theorem deduce

$$K_f = mgd\sin\theta - \mu_k mgd\cos\theta. \tag{7.20}$$

This leads to $v_f = 6.86\,\text{m/s}$.

- Observe that the final velocity is independent of mass m.

Lecture-Example 7.5:
A mass undergoes uniform circular motion, that is, it moves along a circle at constant speed.

- What is the work done by the net force on the mass? (Hint: Determine the direction of the acceleration of the mass at a particular instant? Determine the direction of the net force acting on the mass at this instant? Determine the direction of displacement at this particular instant?)

- What is the change in the kinetic energy of the mass, while it goes around the circle three times?

7.3 Conservative forces and potential energy

The work done by a conservative force is independent of the path taken by the mass. Thus, the work done by a conservative force is completely determined by the initial and final position of the mass. That is, the work done by the force is conveniently defined as the negative change in potential energy U associated with the conservative force,

$$W = \int_{\vec{\mathbf{r}}_i}^{\vec{\mathbf{r}}_f} d\vec{\mathbf{r}} \cdot \vec{\mathbf{F}} = -\Delta U. \tag{7.21}$$

The work-energy theorem, with emphasis on this distinction, is

$$(W_1^{\text{nc}} + W_2^{\text{nc}} + \ldots) + (W_1^{\text{c}} + W_2^{\text{c}} + \ldots) = \Delta K, \tag{7.22}$$

where 'nc' in superscript stands for non-conservative force and 'c' in superscript stands for conservative force. It is then expressed in the form

$$(W_1^{\text{nc}} + W_2^{\text{nc}} + \ldots) = \Delta K + (\Delta U_1 + \Delta U_2 + \ldots). \tag{7.23}$$

Thus, if there are no non-conservative forces acting on the system, the change in energy of the system is independent of the path and is completely determined by the initial and final positions.

Gravitational potential energy

The force of gravity is a conservative force. The work done by the gravitational force is completely determined by the change in height of the mass m,

$$W_g = -mg\Delta y = -\Delta U_g, \tag{7.24}$$

where $\Delta y = y_f - y_i$. It depends only on the initial and final heights. Thus, it is conveniently expressed in terms of the gravitational potential energy function

$$U_g = mgy. \tag{7.25}$$

Lecture-Example 7.6:

- Determine the work done by force of gravity in the following processes.
  1. A person lifts a $m = 3.0\,\text{kg}$ block a vertical distance $h = 10.0\,\text{m}$ and then carries the block horizontally a distance $x = 50.0\,\text{m}$.
  2. A person carries the block horizontally a distance $x = 50.0\,\text{m}$ and then lifts it a vertical distance $h = 10.0\,\text{m}$
  3. A person carries the block along the diagonal line.
- Observe that the work done by the force of gravity is independent of the path. Observe that the work done by force of gravity is zero along a closed path. Observe that the force of gravity does not do any work while moving horizontally. An arbitrary path can be broken into vertical and horizontal sections, which corresponds to path independence.

Lecture-Example 7.7:
A mass of $m = 25.0\,\text{kg}$ slides down a *frictionless* incline that makes an angle of $\theta = 30.0°$ with the horizontal. Assume that the mass starts from rest. The two forces acting on the mass during the slide are the normal force and the force of gravity. The mass slides $d = 10.0\,\text{m}$ along the incline.

- Work-energy theorem states
$$W_N + W_g = \Delta K. \tag{7.26}$$
The work done by the normal force is zero,
$$W_N = 0. \tag{7.27}$$
The work done by the force of gravity on the mass is
$$W_g = mgd\cos(90-\theta) = mgd\sin\theta = 1225\,\text{J}. \tag{7.28}$$

- The change in gravitational potential energy is
$$\Delta U_g = -W_g = -1225\,\text{J}. \tag{7.29}$$
Since $W_N = 0$, the change in kinetic energy of the mass is equal to the work done by the force of gravity,
$$\Delta K = W_g = 1225\,\text{J}. \tag{7.30}$$
The velocity of the mass at the end of the slide is then determined to be $9.90\,\text{m/s}$.

Lecture-Example 7.8: (Roller coaster)
A roller coaster of mass $m = 500.0\,\text{kg}$ moves on the curve described in Figure 7.4. Assume frictionless surface. It starts from rest, $v_A = 0\,\text{m/s}$ at point A height $h_A = 40.0\,\text{m}$.

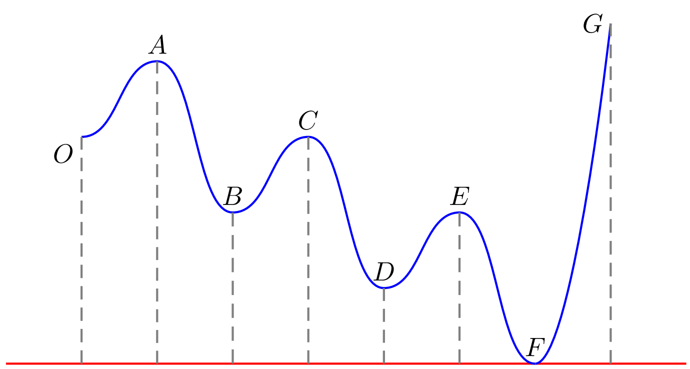

Figure 7.4: Lecture-Example 7.8.

- Work-energy theorem states
$$W_N + W_g = \Delta K. \tag{7.31}$$
Show that the work done by the normal force is zero, $W_N = 0$. Thus, conclude
$$\Delta K + \Delta U_g = 0, \quad \text{or} \quad K_i + U_i = K_f + U_f. \tag{7.32}$$

- Determine the velocity of the mass at points A to G, given $h_B = 20.0\,\text{m}$, $h_C = 30.0\,\text{m}$, $h_D = 10.0\,\text{m}$, $h_E = 20.0\,\text{m}$, $h_F = 0\,\text{m}$, $h_G = 45.0\,\text{m}$. (Answer: See Table 7.1.) Note that the above results are independent of the mass.

- The roller coaster will not reach the point G because it does not have sufficient total energy.

point	h	v	U	K	U+K
A	40.0 m	0 m/s	196 kJ	0 kJ	196 kJ
B	20.0 m	19.8 m/s	98 kJ	98 kJ	196 kJ
C	30.0 m	14.0 m/s	147 kJ	49 kJ	196 kJ
D	10.0 m	24.3 m/s	49 kJ	147 kJ	196 kJ
E	20.0 m	19.8 m/s	98 kJ	98 kJ	196 kJ
F	0 m	28 m/s	0 kJ	196 kJ	196 kJ
G	45.0 m	-	-	-	-

Table 7.1: Lecture-Example 7.8.

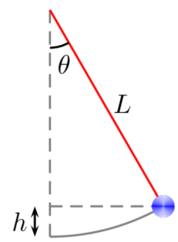

Figure 7.5: Lecture-Example 7.9.

Lecture-Example 7.9:
Figure 7.5 shows a pendulum of length $L = 3.0\,\text{m}$ and mass $m = 5.0\,\text{kg}$. It starts from rest at angle $\theta = 30.0°$. Determine the velocity of the mass when $\theta = 0$.

- Work-energy theorem states
$$W_T + W_g = \Delta K. \tag{7.33}$$
Show that the work done by the tension in the rod is zero,
$$W_T = 0. \tag{7.34}$$
Using $h = L - L\cos\theta$, we have
$$mgh_i + K_i = mgh_f + K_f. \tag{7.35}$$
- How much work does its weight do on the ball?
- What is the change in the gravitational potential energy of the ball Earth system?
- What is the kinetic energy of the ball at its lowest point?
- What is the velocity of the ball at its lowest point?
- If mass m were doubled, would the velocity of the ball at its lowest point increase, decrease, or remain same?

Elastic potential energy of a spring

Elastic materials, for example a spring, when stretched exhibit a restoring force in the opposite direction of the stretch. This is stated as Hooke's law,
$$F = -kx, \tag{7.36}$$

where for the case of springs k is a material dependent quantity called the spring constant. The work done by an elastic force is

$$W_s = \int_{x_i}^{x_f} (-kx)dx = -\left(\frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2\right). \tag{7.37}$$

Thus, using $W = -\Delta U$, we read out the elastic potential energy function

$$U_s = \frac{1}{2}kx^2. \tag{7.38}$$

Lecture-Example 7.10: (Spring constant)
A mass of 5.0 kg is hung using a spring. At equilibrium the spring is stretched 5.0 cm. Determine the spring constant.

- At equilibrium the force of gravity balances the elastic restoring force,

$$kx = mg. \tag{7.39}$$

  (Answer: $k = 980 \sim 10^3$ N/m.) This could be the spring constant of a spring in a simple weighing scale.

- A car weighing 2000 kg is held by four shock absorbers. Thus, each spring gets a load of 500 kg. At equilibrium if the spring is stretched by 5.0 cm, determine the spring constant of a typical shock absorber. (Answer: $k \sim 10^5$ N/m.)

Lecture-Example 7.11:
A mass m slides down a frictionless incline, starting from rest at point A. After sliding down a distance L (along the incline) it hits a spring of spring constant k at point B. The mass is brought to rest at point C when the spring is compressed by length x. See Figure 7.6.

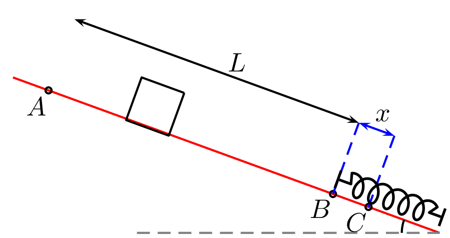

Figure 7.6: Lecture-Example 7.11

- Using work-energy theorem we have

$$W_N + W_g + W_s = \Delta K. \tag{7.40}$$

  Show that the work done by the normal force is zero, $W_N = 0$. Thus, derive

$$K_A + U_A^g + U_A^s = K_B + U_B^g + U_B^s = K_C + U_C^g + U_C^s. \tag{7.41}$$

- Show that the velocity of the mass at point B is given by

$$v_B^2 = 2gL\sin\theta. \tag{7.42}$$

- Show that the maximum compression x in the spring at point C is given by the quadratic equation,

$$x^2 - 2x_0 x - 2x_0 L = 0, \tag{7.43}$$

in terms of the compression x_0 in the spring at equilibrium, given by

$$x_0 = \frac{mg}{k}\sin\theta. \tag{7.44}$$

Thus, we have

$$x = x_0 \pm \sqrt{x_0(x_0 + 2L)}. \tag{7.45}$$

For $x_0 \ll 2L$, show that the solution has the limiting form

$$x \sim \sqrt{2x_0 L}. \tag{7.46}$$

For $2L \ll x_0$, show that the solution has the limiting form $x \sim L$.

Lecture-Example 7.12:
A mass $m = 20.0\,\text{kg}$ slides down a frictionless incline, starting from rest at point A at height $h = 1.0\,\text{m}$. After sliding down the incline it moves horizontally on a frictionless surface before coming to rest by compressing a spring of spring constant $k = 2.0 \times 10^4\,\text{N/m}$ by a length x. See Figure 7.7.

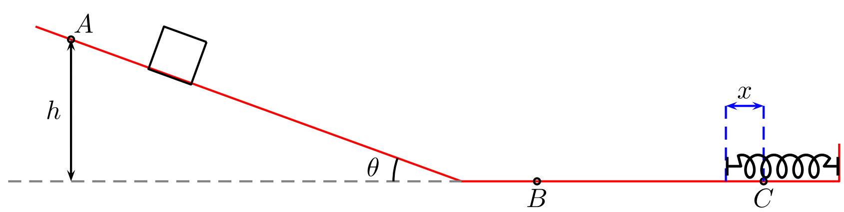

Figure 7.7: Lecture-Example 7.12

- Using work-energy theorem we have

$$W_N + W_g + W_s = \Delta K. \tag{7.47}$$

Show that the work done by the normal force is zero, $W_N = 0$. Thus, derive

$$K_A + U_A^g + U_A^s = K_B + U_B^g + U_B^s = K_C + U_C^g + U_C^s. \tag{7.48}$$

- Determine the velocity of the mass at point B. (Answer: $4.4\,\text{m/s}$.)

- Determine the maximum compression x in the spring. (Answer: $14\,\text{cm}$.)

7.4 Potential energy diagrams

In the absence of non-conservative forces we have

$$\Delta K + \Delta U = 0, \tag{7.49}$$

which allows us to define the total energy as

$$E = \frac{1}{2}mv^2 + U. \tag{7.50}$$

Further, since the work done by conservative forces is independent of the path we can conclude (in one-dimension) that the force is the negative derivative of the potential energy U with respect to position

$$F = -\frac{\partial U}{\partial x}. \tag{7.51}$$

Thus, force is a manifestation of the system trying to minimize its potential energy. In three dimensions we have

$$F = -\vec{\boldsymbol{\nabla}} U = -\hat{\mathbf{i}}\,\frac{\partial U}{\partial x} - \hat{\mathbf{j}}\,\frac{\partial U}{\partial y} - \hat{\mathbf{k}}\,\frac{\partial U}{\partial z}. \tag{7.52}$$

The system is said to be at equilibrium if the force acting on the system is zero. These are called the extremum points in the potential energy profile, where the force (given by the slope) is zero. An extremum point x_0 is a stable point, an unstable point, or a saddle point,

$$\frac{\partial^2 U}{\partial x^2}\bigg|_{x=x_0} \begin{cases} > 0, & \text{(stable point)}, \\ < 0, & \text{(unstable point)}, \\ = 0, & \text{(stable, unstable, or a saddle point)}. \end{cases} \tag{7.53}$$

Lecture-Example 7.13: (Central force)
The potential energy of a particle moving in three dimensions, described by the rectangular coordinates x, y, and z, is given by the function

$$U(x) = \frac{a}{r}, \qquad r = \sqrt{x^2 + y^2 + z^2}, \qquad a > 0. \tag{7.54}$$

- Determine the expression for the force when the particle is at a distance r, at point (x, y, z), from the origin.
- Plot the potential energy with respect to distance r. Plot the magnitude of the force with respect to distance r,
- Is the force attractive (directed towards the origin) or repulsive (directed away from origin)?
- Repeat your analysis for $a < 0$.

Lecture-Example 7.14:
Consider the potential energy curve shown in the figure below.

1. Determine whether the component of force F_x is positive, negative, or zero, at $x = 3\,\text{m}$.
2. Sketch the curve for F_x versus x from $x = 0\,\text{m}$ to $x = 4\,\text{m}$.

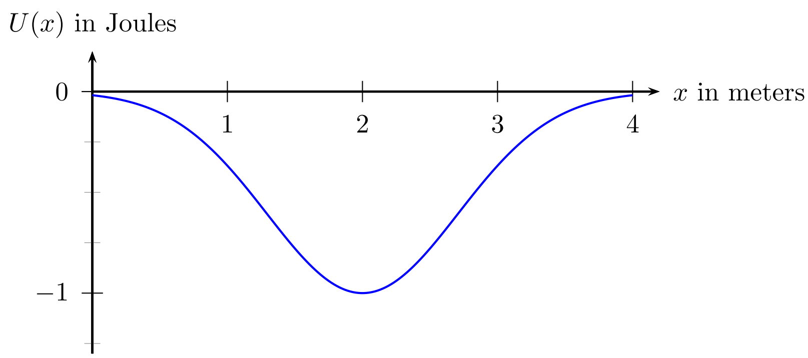

Figure 7.8: Lecture-Example 7.14

Lecture-Example 7.15:
The potential energy of a particle moving along the x axis is given by

$$U(x) = ax^2 - bx^4, \qquad a > 0, \quad b > 0. \tag{7.55}$$

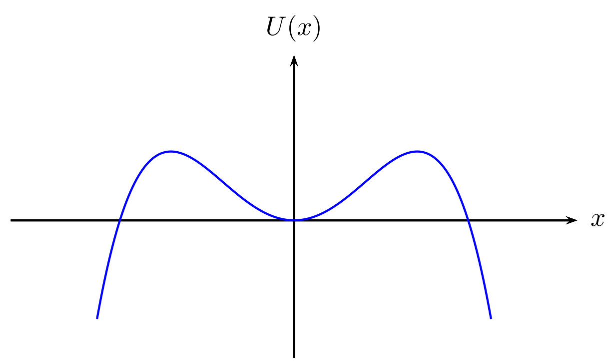

Figure 7.9: Lecture-Example 7.15

- Plot $U(x)$ with respect to x.
- Determine the points on the x axis when the force on the particle is zero, that is, the particle is in equilibrium.
- What can you conclude about the stability of the particle at these points.

Lecture-Example 7.16:

Consider the potential energy curve shown in Figure 7.10 below with respect to distance r, which is given by the expression ($r > 0$)

$$U(r) = \frac{\beta}{2r^2} - \frac{\alpha}{r}, \qquad \alpha > 0, \quad \beta > 0. \tag{7.56}$$

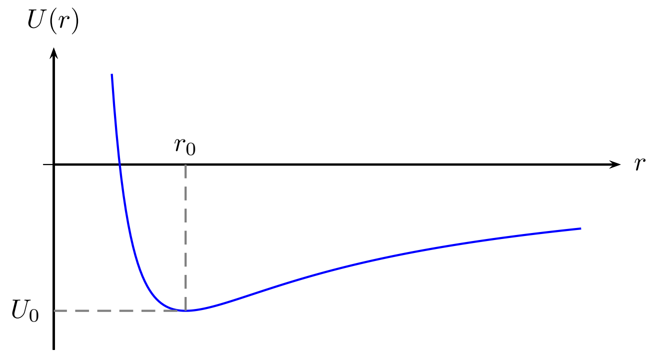

Figure 7.10: Lecture-Example 7.16

- Determine the distance r_0 at which the force corresponding to this potential energy is zero.
- Determine the potential energy U_0 when the force is zero.

Chapter 8

Work and energy (contd.)

This chapter has been created to sync with the numbering of the assigned textbook. The contents of this chapter in the textbook has been covered in the earlier chapter here.

Chapter 9

Collisions: Linear momentum

9.1 Momentum

Using the definition of momentum,

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}}, \tag{9.1}$$

Newton's laws can be expressed in the form

$$\vec{\mathbf{J}}_1 + \vec{\mathbf{J}}_2 + \ldots = \Delta\vec{\mathbf{p}}, \tag{9.2}$$

where

$$\vec{\mathbf{J}}_i = \int_{t_i}^{t_f} \vec{\mathbf{F}}_i dt \tag{9.3}$$

is the impulse due to force $\vec{\mathbf{F}}_i$.

Lecture-Example 9.1: When a ball of mass $m_1 = 1.00\,\text{kg}$ is falling (on Earth of mass $m_2 = 5.97 \times 10^{24}\,\text{kg}$) what are the individual accelerations of the ball and Earth?

Lecture-Example 9.2: A student of mass $m = 60.0\,\text{kg}$ jumps off a table at height $h = 1.00\,\text{m}$. While hitting the floor he bends his knees such that the time of contact is 100.0 ms. What is the force exerted by the floor on the student? If the student does not bend his knees the time of contact is 10.0 ms. What is the new force exerted by the floor now? (Answer: 2660 N versus 26600 N.)

Lecture-Example 9.3: A drop of rain and a pellet of hail, of the same mass $m = 1.00\,\text{g}$, hits the roof of a car with the same speed $v = 5.00\,\text{m/s}$. Rain drop being liquid stays in contact with the roof for 100.0 ms, while hail being solid rebounds (assume with same speed $v = 5.00\,\text{m/s}$) and thus stays in contact for a mere 1.00 ms. Calculate the force exerted by each on the roof of the car. (The numbers quoted here are based on reasonable guesses, and could be off by an order of magnitude.)

9.2 Conservation of linear momentum

If the net external force on a system is zero the change in momentum is zero, or the momentum is conserved. In a collision involving two masses we can write

$$\vec{\mathbf{F}}_1^{\text{ext}} + \vec{\mathbf{C}}_{12} = \frac{d\vec{\mathbf{p}}_1}{dt}, \tag{9.4}$$

$$\vec{\mathbf{F}}_2^{\text{ext}} + \vec{\mathbf{C}}_{21} = \frac{d\vec{\mathbf{p}}_2}{dt}, \tag{9.5}$$

where $\vec{\mathbf{C}}_{12}$ and $\vec{\mathbf{C}}_{21}$ are contact forces, which are action-reaction pairs that are equal and opposite in directions. If the external forces add up to to zero there is no change in momentum and we have the conservation of linear momentum

$$\vec{\mathbf{p}}_{1i} + \vec{\mathbf{p}}_{2i} = \vec{\mathbf{p}}_{1f} + \vec{\mathbf{p}}_{2f}. \tag{9.6}$$

9.2.1 Inelastic collisions

Using conservation of linear momentum we have

$$m_1\vec{\mathbf{v}}_{1i} + m_2\vec{\mathbf{v}}_{2i} = m_1\vec{\mathbf{v}}_{1f} + m_2\vec{\mathbf{v}}_{2f}. \tag{9.7}$$

The particular case when the masses entangle together before or after the collision is called a completely inelastic collision.

Lecture-Example 9.4:
A shooter of mass $m_2 = 90.0\,\text{kg}$ shoots a bullet of mass $m_1 = 3.00\,\text{g}$ horizontally, standing on a frictionless surface at rest. If the muzzle velocity of the bullet is $v_{1f} = 600.0\,\text{m/s}$, what is the recoil speed of the shooter? (Answer: $v_{2f} = -2.00\,\text{cm/s}$.)

Lecture-Example 9.5:
A shooter of mass $m_2 = 90.0\,\text{kg}$ shoots a bullet of mass $m_1 = 3.00\,\text{g}$ in a direction $\theta = 60.0°$ with respect to the horizontal, standing on a frictionless surface at rest. If the muzzle velocity of the bullet is $v_{1f} = 600.0\,\text{m/s}$, what is the recoil speed of the shooter? (Answer: $v_{2f} = -1.00\,\text{cm/s}$.)

Lecture-Example 9.6: (Ballistic pendulum)
A bullet with mass $m_1 = 3.00\,\text{g}$ is fired into a wooden block of mass $m_2 = 1.00\,\text{kg}$, that hangs like a pendulum. The bullet is embedded in the block (complete inelastic collision). The block (with the bullet embedded in it) goes $h = 30.0\,\text{cm}$ high after collision. Calculate the speed of the bullet before it hit the block.

Lecture-Example 9.7: (Collision of automobiles at an intersection.)
A car of mass $m_1 = 2000.0\,\text{kg}$ is moving at speed $v_{1i} = 20.0\,\text{m/s}$ towards East. A truck of mass $m_2 = 5000.0\,\text{kg}$ is moving at speed $v_{2i} = 10.0\,\text{m/s}$ towards North. They collide at an intersection and get entangled (complete inelastic collision). What is the magnitude and direction of the final velocity of the entangled automobiles?

- Repeat the calculation for a semi-truck (ten times heavier) moving at the same speed.

9.2.2 Elastic collisions in 1-D

In an elastic collision, in addition to momentum being conserved, the kinetic energy is also conserved. This requires no loss of energy in the form of sound and heat. Conservation of kinetic energy leads to

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2. \tag{9.8}$$

In conjunction with the conservation of momentum,

$$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f}, \tag{9.9}$$

this leads to the corollary

$$v_{1i} + v_{1f} = v_{2i} + v_{2f}. \tag{9.10}$$

Together we can solve for the final velocities:

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)v_{2i}, \tag{9.11a}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_{2i}. \tag{9.11b}$$

Consider the following cases:

1. $m_1 = m_2$: Implies swapping of velocities!

2. $v_{2i} = 0$:

3. $v_{2i} = 0$, $m_1 \ll m_2$:

4. $v_{2i} = 0$, $m_1 \gg m_2$:

Lecture-Example 9.8: A mass $m_1 = 1.00\,\text{kg}$ moving with a speed $v_{1i} = +10.0\,\text{m/s}$ (elastically) collides with another mass $m_2 = 1.00\,\text{kg}$ initially at rest. Describe the motion after collision. (Answer: $v_{1f} = 0\,\text{m/s}$ and $v_{2f} = -v_{1i} = +10.0\,\text{m/s}$.)

Lecture-Example 9.9: A mass $m_1 = 1.00\,\text{kg}$ moving with a speed $v_{1i} = +10.0\,\text{m/s}$ (elastically) collides with another mass $m_2 = 100.0\,\text{kg}$ initially at rest. Describe the motion after collision. (Answer: $v_{1f} = -9.80\,\text{m/s}$ and $v_{2f} = +0.198\,\text{m/s}$.)

Lecture-Example 9.10: A mass $m_1 = 100\,\text{kg}$ moving with a speed $v_{1i} = +10\,\text{m/s}$ (elastically) collides with another mass $m_2 = 1\,\text{kg}$ initially at rest. Describe the motion after collision. (Answer: $v_{1f} = +9.80\,\text{m/s}$ and $v_{2f} = +19.8\,\text{m/s}$.)

Lecture-Example 9.11: (Rebounce of tennis ball on basketball.)
A tennis ball of mass $m_1 = 60.0\,\text{g}$ is dropped with a basketball of mass $m_2 = 0.600\,\text{kg}$ from a height of $h = 1\,\text{m}$. How high does the tennis ball return back?

Lecture-Example 9.12: An electron collides elastically with a stationary hydrogen atom. The mass of the hydrogen atom is 1837 times that of the electron. Assume that all motion, before and after the collision, occurs along the same straight line. What is the ratio of the kinetic energy of the hydrogen atom after the collision to that of the electron before the collision?

Using Eqs. (9.11) for elastic collisions in 1-D, with $m_2 = 1837 m_1$ and $v_{2i} = 0$, obtain

$$\frac{v_{2f}}{v_{1i}} = \frac{2}{1838}. \tag{9.12}$$

Then, we have the ratio

$$\frac{K_{2f}}{K_{1i}} = \frac{m_2}{m_1}\left(\frac{v_{2f}}{v_{1i}}\right)^2 = 1837\left(\frac{2}{1838}\right)^2 \sim \frac{1}{459.8}. \tag{9.13}$$

9.3 Center of mass

The center of mass of a distribution of mass (in one dimension) is defined as

$$x_{\text{cm}} = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i} = \frac{\int x\, dm}{\int dm}. \tag{9.14}$$

In the language of statistics, center of mass is the first moment of mass. The total mass itself is the zeroth moment of mass. The term weighted average is based on this concept. In three dimensions the center of mass of a distribution of mass is defined as

$$\vec{\mathbf{r}}_{\text{cm}} = \frac{\sum_{i=1}^{N} m_i \vec{\mathbf{r}}_i}{\sum_{i=1}^{N} m_i} = \frac{\int \vec{\mathbf{r}}\, dm}{\int dm}. \tag{9.15}$$

Lecture-Example 9.13: (Meter stick)
A uniform meter stick has a mass $m_1 = 10.0\,\text{kg}$ placed at 100.0 cm mark and another mass $m_2 = 20.0\,\text{kg}$ placed at 20.0 cm mark. Determine the center of mass of the stick and the two masses together. (Answer: $x_{\text{cm}} = 46.7\,\text{cm}$.)

Lecture-Example 9.14: (Earth-Moon)
Determine the center of mass of the Earth-Moon system. In particular, determine if the center of mass of the Earth-Moon system is inside or outside the Earth. Given mass of Earth is 81 times the mass of Moon, and the distance between the center of Earth and center of Moon is 60 times the radius of Earth. Or, given the masses $M_{\text{Earth}} = 5.97 \times 10^{24}\,\text{kg}$, $M_{\text{Moon}} = 7.35 \times 10^{22}\,\text{kg}$, the radiuses $R_{\text{Earth}} = 6.37 \times 10^{6}\,\text{m}$, $R_{\text{Moon}} = 1.74 \times 10^{6}\,\text{m}$, and the distance between them is $r = 384 \times 10^{6}\,\text{m}$. (Answer: $4.67 \times 10^{6}\,\text{m}$ from the center of Earth on the line passing through the centers of Earth and Moon.)

Lecture-Example 9.15:
Three masses are placed on a plane such that the coordinates of the masses are, $m_1 = 1.0\,\text{kg}$ at $(1, 0)$, $m_2 = 2.0\,\text{kg}$

at $(2,0)$, and $m_3 = 3.0\,\text{kg}$ at $(0,3)$. Determine the coordinates of the center of the mass of the three masses. (Answer: $(\frac{5}{6}, \frac{3}{2})$.)

Lecture-Example 9.16: (Rod of uniform density)
An infinitely thin rod of length L and mass M has a uniform mass per unit length

$$\lambda = \frac{M}{L} = \frac{dm}{dx}. \tag{9.16}$$

Measuring x from one end of the rod show that

$$x_{\text{cm}} = \frac{L}{2}. \tag{9.17}$$

Lecture-Example 9.17: (Rod with non-uniform mass density)
An infinitely thin rod of length L has a mass per unit length described by

$$\lambda = \frac{dm}{dx} = ax, \tag{9.18}$$

where x is measured from one end of the rod. Show that the center of mass of the rod is

$$x_{\text{cm}} = \frac{2}{3}L. \tag{9.19}$$

Further, deduce that

$$a = \frac{2M}{L^2}. \tag{9.20}$$

Chapter 10

Rotational motion

10.1 Vector product

Vector product (or the cross product) of two vectors

$$\vec{\mathbf{A}} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}, \tag{10.1a}$$
$$\vec{\mathbf{B}} = B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}} + B_z\,\hat{\mathbf{k}}, \tag{10.1b}$$

is given by

$$\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_yB_z - A_zB_y)\,\hat{\mathbf{i}} - (A_xB_z - A_zB_x)\,\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\,\hat{\mathbf{k}} \tag{10.2a}$$
$$= AB\sin\theta\,\hat{\mathbf{n}} \tag{10.2b}$$

where θ is the angle between the two vectors. The vector product measures the area associated with the two vectors. The direction of the vector product $\vec{\mathbf{C}}$ is given by the right-hand rule. The right-hand rule is a mnemonic that associates the thumb to the vector $\vec{\mathbf{A}}$, the fingers to the vector $\vec{\mathbf{B}}$, such that the vector $\vec{\mathbf{C}}$ is in the direction facing the palm of the right hand.

In discussions concerning three dimensions we often have quantities pointing in and out of a plane. We shall use the notation $\odot$ to represent a direction coming out of the plane, and $\otimes$ to represent a direction going into the plane. As a mnemonic one associates the dot with the tip of an arrow coming out of the page and the cross with the feathers of an arrow going into the page.

10.2 Rotational kinematics

A rigid object will be defined as an object with the constarint that the relative distances of any two points inside the body does not vary with time. We will confine our attention to rotational motion of rigid bodies about a fixed axis. Thus, the motion of a particle is confined to a plane perpendicular to the axis.

Since the distance of a point from the axis remains fixed for a rigid body, we can specify the motion of this point with respect to the axis by specifying the angle it is rotated. The infinitismal angular displacement is defined as the vector $d\vec{\boldsymbol{\theta}}$ whose direction specifies the axis of rotation and the magnitude specifies the amount of rotation about the axis. The change in position of the point due to this rotation is given by

$$d\vec{\mathbf{r}} = d\vec{\boldsymbol{\theta}} \times \vec{\mathbf{r}}, \tag{10.3}$$

which for the case of rigid rotation is simply

$$dr = d\theta\, r. \tag{10.4}$$

We immediately have the relation

$$\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}, \tag{10.5}$$

in terms of the instantaneous angular velocity

$$\vec{\boldsymbol{\omega}} = \frac{d\vec{\boldsymbol{\theta}}}{dt}, \tag{10.6}$$

which for the case of rigid rotation is simply

$$v = \omega\, r. \tag{10.7}$$

Differentiating the velocity we obtain

$$\vec{\mathbf{a}} = \vec{\boldsymbol{\alpha}} \times \vec{\mathbf{r}} + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}), \tag{10.8}$$

in terms of the instantaneous angular acceleration

$$\vec{\boldsymbol{\alpha}} = \frac{d\vec{\boldsymbol{\omega}}}{dt}, \tag{10.9}$$

which for the case of rigid rotation corresponds to the tangential and radial accelerations

$$a_T = \alpha\, r \quad \text{and} \quad a_r = -\omega^2 r \tag{10.10}$$

respectively.

Rotation motion with constant angular acceleration

For the case of rotation motion with constant angular acceleration the angular velocity and the angular acceleration are given by

$$\frac{\Delta\theta}{\Delta t} = \frac{\omega_f + \omega_i}{2}, \tag{10.11a}$$

$$\alpha = \frac{\omega_f - \omega_i}{\Delta t}. \tag{10.11b}$$

Eqs. (10.11a) and (10.11b) are two independent equations involving five independent variables: $\Delta t, \Delta\theta, \omega_i, \omega_f, \alpha$. We can further deduce,

$$\Delta\theta = \omega_i \Delta t + \frac{1}{2}\alpha \Delta t^2, \tag{10.11c}$$

$$\Delta\theta = \omega_f \Delta t - \frac{1}{2}\alpha \Delta t^2, \tag{10.11d}$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta, \tag{10.11e}$$

obtained by subtracting, adding, and multiplying, Eqs. (10.11a) and (10.11b), respectively.

Lecture-Example 10.1:
Starting from rest a wheel rotates with uniform angular acceleration $3.0\,\text{rad/s}^2$. Determine the instantaneous angular velocity of the wheel after $3.0\,\text{s}$.

Lecture-Example 10.2:
The angular position of a point on the rim of a rotating wheel is given by $\theta = 4.0\,t - 2.0\,t^2 + t^3$, where θ is in radians and t is in seconds.

- Determine the angular velocity at $t = 6.0\,\text{s}$.
- Determine the instantaneous angular acceleration at $t = 6.0\,\text{s}$.

10.3 Torque

The ability of a force to contribute to rotational motion, about an axis, is measured by torque

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}. \tag{10.12}$$

Lecture-Example 10.3:
A force of 10.0 N is exerted on a door in a direction perpendicular to the plane of the door at a distance of 40.0 cm from the hinge. Determine the torque exerted by the force. (Answer: 4.00 Nm.)

10.4 Moment of inertia

For a particle rotating 'rigidly' about an axis, the tendency to be in the state of rotational rest or constant angular velocity, the rotational inertia, is given the moment of inertia

$$I = mr^2, \tag{10.13}$$

where r is the perpendicular distance of the mass m to the axis. The moment of inertia of a distribution of mass is given by

$$I = \sum_{i=1}^{N} m_i r_i^2 = \int r^2\, dm. \tag{10.14}$$

In the language of statistics, moment of inertia is the second moment of mass.

Lecture-Example 10.4: (Rotational inertia)
A *massless* rod is hinged so that it can rotate about one of its ends. Masses $m_1 = 1.0\,\text{kg}$ and $m_2 = 20.0\,\text{kg}$ are attached to the rod at $r_1 = 1.0\,\text{m}$ and $r_2 = 5.0\,\text{cm}$ respectively. Determine the moment of inertia of the configuration. (Answer: $1.1\,\text{kg}\cdot\text{m}^2$.)

- Repeat the calculation for $r_2 = 0.5\,\text{m}$. (Answer: $6.0\,\text{kg}\cdot\text{m}^2$.)

Lecture-Example 10.5: (Uniform rod)
Determine the moment of inertia of an infinitely thin rod of mass M and length L, when the axis is perpendicular to the rod and passing through the center of the rod. (Answer: $I = ML^2/12$.) Repeat for the case when the axis is perpendicular to the rod and passing through on of the ends of the rod. (Answer: $I = ML^2/3$.)

Lecture-Example 10.6: (Comparing moment of inertia)
Show that

$$I_{\substack{\text{solid sphere}\\ \text{about diameter}}} < I_{\substack{\text{solid cylinder}\\ \text{about axis}}} < I_{\substack{\text{spherical shell}\\ \text{about diameter}}} < I_{\substack{\text{cylindrical shell}\\ \text{about axis}}}. \tag{10.15}$$

10.5 Rotational dynamics

For the case when the moment of of inertia I of a body does not change in time, the rotational dynamics is described by the equation

$$\vec{\boldsymbol{\tau}}_1 + \vec{\boldsymbol{\tau}}_2 + \ldots = I\vec{\boldsymbol{\alpha}}. \tag{10.16}$$

Lecture-Example 10.7:
A uniform solid cylinder of radius R and mass M is free to rotate about its symmetry axis. The cylinder acts like a pulley. A string wound around the cylinder is connected to a mass m, which falls under gravity. See Fig. 10.1. What is the angular acceleration α of the cylinder?

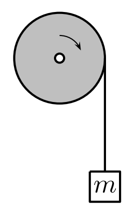

Figure 10.1: Lecture-Example 10.7

- Using Newton's law, for the mass m, show that

$$mg - T = ma, \tag{10.17}$$

  where T is the tension in the string. Using the Newton's law for torque, for the mass M, deduce the relation

$$T = \frac{1}{2} MR\alpha. \tag{10.18}$$

  Presuming the string does not stretch and rolls the cylinder perfectly we also have the constraint

$$a = \alpha R. \tag{10.19}$$

- Determine the acceleration a of the mass m to be

$$a = \frac{m}{\left(m + \frac{M}{2}\right)} g. \tag{10.20}$$

  Determine the angular acceleration α of the cylinder, and the tension T in the string.

10.6 Rotational work-energy theorem

The rotational work-energy theorem states

$$W_1 + W_2 + \ldots = K_{\text{rot}}, \tag{10.21}$$

where the work done by the torque $\vec{\boldsymbol{\tau}}_i$ is given by

$$W_i = \int_i^f \vec{\boldsymbol{\tau}}_i \cdot d\vec{\boldsymbol{\theta}} \tag{10.22}$$

and the rotational kinetic energy is given by

$$K_{\text{rot}} = \frac{1}{2} I\omega^2. \tag{10.23}$$

Lecture-Example 10.8:
A solid sphere and a spherical shell, both of same mass M and same radius R, start from rest at a height h on an incline.

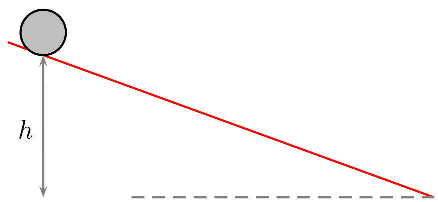

Figure 10.2: Lecture-Example 10.8

- Using the translational work-energy theorem show that

  $$mgh - F_f d = \frac{1}{2} M v^2, \tag{10.24}$$

  where F_f is the force of friction and d is the distance along the incline. Using rotational work-energy theorem show that

  $$F_f R\theta = \frac{1}{2} I\omega^2, \tag{10.25}$$

  where θ is the angular displacement corresponding to the distance d. For rolling without slipping or sliding argue that

  $$d = R\theta \qquad \text{and} \qquad v = \omega R. \tag{10.26}$$

  Verify that, for rolling motion, there is no work done by the force of friction. That is, for rolling motion, the translational work done by the force of friction exactly cancels the rotational work done by the force of friction. Thus, deduce the relation

  $$mgh = \frac{1}{2} M v^2 + \frac{1}{2} I\omega^2. \tag{10.27}$$

  Determine the velocity of the solid sphere and the spherical shell to be

  $$v = \sqrt{\frac{2gh}{1+p}}, \tag{10.28}$$

  where we defined $I = pMR^2$.

- Using translational Newton's law show that

  $$mg \sin\theta - F_f = ma, \tag{10.29}$$

  and using rotational Newton's law show that

  $$F_f R = I\alpha. \tag{10.30}$$

  Thus, deduce the relation

  $$a = \frac{g \sin\theta}{1+p}. \tag{10.31}$$

- Model a raw egg as a spherical shell and a boiled egg as a solid sphere, and deduce which of them will roll down the incline faster.

10.7 Direction of friction on wheels

Consider an illustrative example consisting of a simple two-wheeler. It consists of two wheels connected by a rod. The front wheel is driven by a torque provided by an engine, and the rear wheel is pulled forward by the rod. Thus, the front wheel is driven by a torque, and the rear wheel is driven by a force. Let us assume perfect rolling, with no slipping or sliding. Let the front wheel have radius R_1, mass m_1, and moment of inertia $I_1 = p_1 m_1 R_1^2$, and let the rear wheel have radius R_2, mass m_2, and moment of inertia $I_2 = p_2 m_2 R_2^2$,

Accelerating forward while moving forward

Friction acts in the forward direction on the front wheel, and in the backward direction on the rear wheel. Note that F_{pull} is the same on both the tires because of Newton's third law.

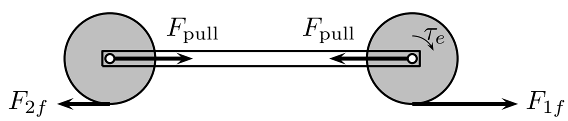

Figure 10.3: A simple two-wheeler accelerating forward.

$$\tau_e - R_1 F_{1f} = I_1 \alpha_1 \quad \text{(Torque equation for front wheel),} \tag{10.32a}$$
$$F_{1f} - F_{\text{pull}} = m_1 a_1 \quad \text{(Force equation for front wheel),} \tag{10.32b}$$
$$R_2 F_{2f} = I_2 \alpha_2 \quad \text{(Torque equation for rear wheel),} \tag{10.32c}$$
$$F_{\text{pull}} - F_{2f} = m_2 a_2 \quad \text{(Force equation for rear wheel).} \tag{10.32d}$$

For rolling motion, we have the constraints,

$$a_1 = \alpha_1 R_1 \quad \text{and} \quad a_1 = \alpha_1 R_1, \tag{10.33}$$

and the rigidity of the two-wheel configuration further requires the constraint

$$a_1 = a_2 = a. \tag{10.34}$$

Thus, we can derive the acceleration of the two-wheeler using

$$a = \frac{\tau_e}{R_1 \big[(1 + p_1) m_1 + (1 + p_2) m_2\big]}. \tag{10.35}$$

In terms of the acceleration of the system we can determine all other forces. For the particular case $R_1 = R_2 = R$, $m_1 = m_2 = m$, and $p_1 = p_2 = 1/2$,

$$ma = 2F_{2f} = \frac{2}{3} F_{\text{pull}} = \frac{2}{5} F_{1f} = \frac{1}{3} \frac{\tau_e}{R}. \tag{10.36}$$

Constant speed

What happens if the engine was switched off? Using Eq. (10.36) we learn that this requires the acceleration a to be zero, which immediately implies that all the forces are zero in this case. This is unphysical, and is a consequence of the extreme constraint imposed by perfect rolling.

To understand this, consider a single wheel rolling forward under the influence of friction alone. If the friction is assumed to be acting in the forward direction it will lead to translational acceleration, with angular decceleration of the wheel, which is possible simultaneaously only when imperfect rolling is allowed, or when frictional force is zero.

Accelerating backward (decceleerating) while moving forward

Next, let a torque τ_b be applied, using brakes, on the front wheel. Friction acts in the backward direction on the front wheel, and in the forward direction on the rear wheel.

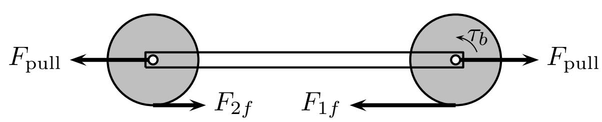

Figure 10.4: A simple two-wheeler deccelerating.

$$-\tau_b + R_1 F_{1f} = -I_1 \alpha_1 \qquad \text{(Torque equation for front wheel),} \qquad (10.37a)$$
$$-F_{1f} + F_{\text{pull}} = -m_1 a_1 \qquad \text{(Force equation for front wheel),} \qquad (10.37b)$$
$$-R_2 F_{2f} = -I_2 \alpha_2 \qquad \text{(Torque equation for rear wheel),} \qquad (10.37c)$$
$$-F_{\text{pull}} + F_{2f} = -m_2 a_2 \qquad \text{(Force equation for rear wheel).} \qquad (10.37d)$$

Thus, we can derive the acceleration of the two-wheeler using

$$a = \frac{\tau_b}{R_1\big[(1+p_1)m_1 + (1+p_2)m_2\big]}. \qquad (10.38)$$

This leads to the same magnitudes for the forces as in the case of forward acceleration.

Lecture-Example 10.9: (Comments)
The key observation is that the friction on the front wheel opposes the torque and the friction on the rear wheel opposes the force. In more complicated systems a wheel is acted on by torques and forces simultaneously, and in such situations the friction opposes either the torque or the force at a given moment in time.

- Which tires (front or rear) wears more due to friction?
- For perfect rolling verify that the total work done by the force of friction is zero.
- Derive the above expressions for the case when the system is accelerating backward by applying brakes in the rear wheel alone.

10.8 Homework problems

Homework-Problem 10.1:
A potter's wheel moves uniformly from rest to an angular speed of 0.19 rev/s in 34 s.

1. Find its angular acceleration in radians per second per second.
2. Would doubling the angular acceleration during the given period have doubled final angular speed?

Hints:

1. Following is given: initial angular speed $\omega_i = 0$, final angular speed $\omega_f = 0.19\,\text{rev/s}$, time elapsed $\Delta t = 34\,\text{s}$. Convert rev/s to rad/s using the conversion 1 revolution = 1 radian. Use the appropriate equation from Eqs. (10.11) to find the angular acceleration α. (Answer: $\alpha = 0.035\,\text{rad/s}^2$.)
2. For the case when the initial angular speed is zero, we have $\omega_f = \alpha \Delta t$. Using this equation analyse what happens to to the final angular speed ω_f when the angular acceleration α is doubled.

Homework-Problem 10.2:
A wheel starts from rest and rotates with constant angular acceleration to reach an angular speed of 11.2 rad/s in 2.90 s.

1. Find the magnitude of the angular acceleration of the wheel.
2. Find the angle in radians through which it rotates in this time interval.

Hints:

1. Following is given: initial angular speed $\omega_i = 0$, final angular speed $\omega_f = 11.2\,\text{rad/s}$, time elapsed $\Delta t = 2.90\,\text{s}$. Use the appropriate equation from Eqs. (10.11) to find the angular acceleration α. (Answer: $\alpha = 3.86\,\text{rad/s}^2$.)
2. Following is given: initial angular speed $\omega_i = 0$, final angular speed $\omega_f = 11.2\,\text{rad/s}$, time elapsed $\Delta t = 2.90\,\text{s}$. Use the appropriate equation from Eqs. (10.11) to find the angular displacement $\Delta\theta$. (Answer: $\Delta\theta = 16.2\,\text{rad}$.)

Homework-Problem 10.3:
An electric motor rotating a workshop grinding wheel at $1.02 \times 10^2\,\text{rev/min}$ is switched off. Assume the wheel has a constant negative angular acceleration of magnitude $1.92\,\text{rad/s}^2$.

1. How long does it take the grinding wheel to stop?
2. Through how many radians has the wheel turned during the time interval found in earlier part?

Hints:

1. Following is given: initial angular speed $\omega_i = 1.02 \times 10^2$ rev/min, final angular speed $\omega_f = 0$, angular acceleration $\alpha = -1.92\,\text{rad/s}^2$. Convert rev/min to rad/s, use the conversion 1 revolution = 1 radian. Use the appropriate equation from Eqs. (10.11) to find the alapsed time Δt. (Answer: $\Delta t = 5.56\,\text{s}$.)

2. Following is given: initial angular speed $\omega_i = 1.02 \times 10^2$ rev/min, final angular speed $\omega_f = 0$, angular acceleration $\alpha = -1.92\,\text{rad/s}^2$. You also know the elapsed time Δt from the earlier part. Use any equation from Eqs. (10.11) containing the angular displacement $\Delta\theta$ to find the angular displacement. (Answer: $\Delta\theta = 29.7\,\text{rad}$.)

Part II

Electricity and Magnetism

Chapter 23

Electric force and electric field

23.1 Electric charge

Like mass is a fundamental property of an object, electric charge is another fundamental property of an object or a particle. Unlike mass, which is always non-negative, charge can be positive or negative. Charge is measured in units of Coulomb.

1. Electric charge is always conserved.

2. Electric charge is quantized. That is, it always comes in integer multiples of a fundamental charge

$$e \sim 1.60 \times 10^{-19}\,\text{C}. \tag{23.1}$$

   It is instructive to compare the electric charge and mass of the three particles that constitutes all atoms.

3. All macroscopic objects get their charge from the electrons and protons that constitute them. Charges are not always free to move inside an object. We will often consider two extremes: A perfect conductor in which the charges are completely free to move, and a perfect insulator in which the charges are static. Metals (like gold and copper) are pretty much perfect conductors, and wood and rubber are close to perfect insulators. Vacuum is the perfect insulator.

To get an an insight regarding the amount of charge contained in a Coulomb of charge we list a few typical charges we encounter in Table 23.1.

Lecture-Example 23.1:
Determine the number of electrons in one gram of electron. Then calculate the total charge of one gram of electron.

- One gram of electron has about 10^{27} electrons, and a total charge of about $10^{8}\,\text{C}$, an enormous amount of charge.

Particle	Charge	Mass
Electron	$-e$	$\sim 9.109 \times 10^{-31}\,\text{kg}$
Proton	$+e$	$\sim 1.672 \times 10^{-27}\,\text{kg}$
Neutron	0	$\sim 1.674 \times 10^{-27}\,\text{kg}$

Table 23.1: Charge and masses of particles that constitutes all atoms.

10^{-19} C	charge on an electron
10^{-15} C	charge on a typical dust particle
10^{-6} C	this much isolated charge when confined to a region of 10 cm (a typical hand) causes breakdown of air (static electricity).
10^{1} C	this much isolated charge when confined to a region of 1000 m (a typical thundercloud) causes breakdown of air (lightning).
10^{3} C	total charge generated in an alkaline battery. This is not isolated, so does not breakdown air.
10^{6} C	this much isolated charge when confined to a region of 1 m has been predicted to breakdown vacuum.

Table 23.2: Orders of magnitude (charge)

23.2 Coulomb's law

The electrostatic force between two objects with charges q_1 and q_2, separated by distance r, is

$$\vec{\mathbf{F}} = k_e \frac{q_1 q_2}{r^2} \hat{\mathbf{r}}, \tag{23.2}$$

where $\hat{\mathbf{r}}$ encodes the direction content of the force. Like charges repel and unlike charges attract. The constant of proportionality is $k_e \sim 8.99 \times 10^9 \,\text{Nm}^2/\text{C}^2$, which is often expressed in terms of the permittivity of vacuum,

$$\varepsilon_0 \sim 8.85 \times 10^{-12} \frac{\text{C}^2}{\text{Nm}^2}, \qquad \text{using} \qquad k_e = \frac{1}{4\pi\varepsilon_0}. \tag{23.3}$$

Lecture-Example 23.2: (Static electricity)
Consider a neutral balloon of mass $m = 10.0\,\text{g}$ blown up so that it is (aproximately) a sphere of radius $R = 10.0\,\text{cm}$. I can rub it on my shirt so that a certain amount of charge Q is transferred from the balloon to my hand, such that the balloon and my hand acquire unlike charges. We will presume that the charge on the balloon is uniformly distributed on its surface. If the attractive force on the balloon from my hand can barely balance gravity, determine the charge Q. Due to the approximations involved our estimate is expected to be valid to one significant digit only.

- The gravitation force on the balloon is nulled by the electrostatic force,

$$mg = \frac{kQ^2}{R^2}. \tag{23.4}$$

This leads to $Q = 3 \times 10^{-7}\,\text{C}$.

Lecture-Example 23.3: A hydrogen atom consists of an electron orbiting a proton. The radius is about $5.3 \times 10^{-11}\,\text{m}$.

- Find the electrostatic force between an electron and a proton.

$$F_{\text{electric}} = \frac{ke^2}{R^2} \sim 10^{-8}\,\text{N}. \tag{23.5}$$

- Find the gravitational force between an electron and a proton.

$$F_{\text{gravity}} = \frac{Gm_e m_p}{R^2} \sim 10^{-47}\,\text{N}. \tag{23.6}$$

- Find the ratio of the electrostatic force to gravitational force. (This is independent of the radius.)

$$\frac{F_{\text{electric}}}{F_{\text{gravity}}} = \frac{ke^2}{Gm_e m_p} \sim 10^{40}. \tag{23.7}$$

Lecture-Example 23.4: Can we detach the Moon?
If charges of same sign are placed on Earth and Moon it could be possible to negate the gravitational force between them. ($m_{\text{Earth}} \sim 6 \times 10^{24}\,\text{kg}$, $m_{\text{Moon}} \sim 7 \times 10^{22}\,\text{kg}$.) (You do not need the knowledge of the Earth-Moon distance for this calculation, $R \sim 4 \times 10^8\,\text{m}$.)

- Equating the gravitational force to the electrostatic force we have

$$\frac{Gm_E m_{\text{moon}}}{R^2} = \frac{kq^2}{R^2}. \tag{23.8}$$

- Thus, the charge needed to release the Moon is $q = 10^{12}\,\text{C}$, which is about 1 kg of electrons. This is a stupendous amount of charge, which when confined to the volume of Earth will breakdown the atmosphere, though not breakdown vacuum!

Lecture-Example 23.5: Charges $q_1 = +3.0\,\mu\text{C}$ and $q_2 = -1.0\,\mu\text{C}$ are placed a distance $x = 10.0\,\text{cm}$ apart. Presume the two charges to be uniformly spread on identical perfectly conducting spheres of radius $R = 1.0\,\text{cm}$ with masses $m_1 = 100.0\,\text{g}$ and $m_2 = 10\,m_1$.

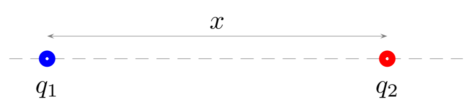

Figure 23.1: Lecture-Example 23.5

- Find the forces $\vec{\mathbf{F}}_{12}$ and $\vec{\mathbf{F}}_{21}$ on the charges. Determine the instantaneous accelerations $\vec{\mathbf{a}}_1$ and $\vec{\mathbf{a}}_2$ of spheres when they are x distance apart. Note that the instantaneous accelerations are not uniform, they are distance dependent and get larger as they get closer. (Answer: $a_1 = 27\,\text{m/s}^2$, $a_2 = 2.7\,\text{m/s}^2$.)

- If let go, the two spheres attract, move towards each other, and come in contact. Once in contact, because the charges are on perfectly conducting spheres, the charges will redistribute on the two spheres. Determine the new charges q_1' and q_2' on the two spheres to be

$$q_1' = q_2' = \frac{q_1 + q_2}{2}. \tag{23.9}$$

  (Answer: $q_1' = q_2' = 1.0\,\mu\text{C}$.)

- Find the repulsive force on the two spheres after they come into contact. Determine the instantaneous accelerations $\vec{\mathbf{a}}_1'$ and $\vec{\mathbf{a}}_2'$ of the two spheres when they are in contact, their centers a distance $2R$ apart. Observe that the smaller mass does most of the movement, relatively. Again, observe that the instantaneous accelerations are not uniform, they are distance dependent and get weaker as they get farther apart. (Answer: $a_1 = 225\,\text{m/s}^2$, $a_2 = 22.5\,\text{m/s}^2$.)

Lecture-Example 23.6: Two positive charges and two negative charges of equal magnitude are placed at the corners of a square of length L, such that like charges are at diagonally opposite corners.

- The magnitude of the force on one of the positive charge is

$$|\vec{\mathbf{F}}_{\text{tot}}| = \left(\sqrt{2} - \frac{1}{2}\right) F_0, \qquad F_0 = \frac{kq^2}{L^2}. \tag{23.10}$$

- Analyze the direction of the force on one of the positive charge.
- If the four charges were free to move, will they collectively move away from each other or move towards each other?

Lecture-Example 23.7: Three identical charges of equal magnitude q are placed at the corners of a triangle of length L. Determine the magnitude of the Coulomb force on one of the charges.

Lecture-Example 23.8: Where is the total force zero?
See Figure 23.2. Two positive charges q_1 and q_2 are fixed to a line. As a multiple of distance D, at what coordinate on the line is the net electrostatic force on a negative charge q_3 zero?

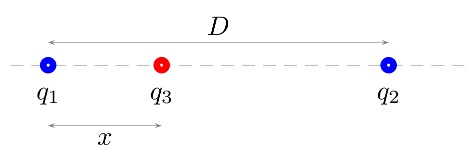

Figure 23.2: Lecture-Example 23.8

- Equate the forces to deduce

$$x = \frac{D}{\left(1 + \sqrt{\frac{q_2}{q_1}}\right)}. \tag{23.11}$$

For $q_2 > q_1$ we have $0 < x < L/2$. And, for $q_2 < q_1$ we have $0 < L/2 < x < L$. In general the equilibrium point is closer to the smaller charge. Investigate if the particle 3 is stable or unstable at this point?

- Repeat the above for a positive charge q_3.
- Repeat the above for unlike q_1 and q_2.

Lecture-Example 23.9: In Figure 23.3, particles 1 and 2 of charge $q_1 = q_2$ are placed on a y axis at distance a from the origin. Particle 3 of charge q_3 is moved gradually along the x axis.

- Show that the electrostatic force on charge q_3 is given by

$$\vec{\mathbf{F}}_{\text{tot}} = \hat{\mathbf{i}}\, kq_1q_3 \frac{2x}{(x^2 + a^2)^{\frac{3}{2}}}. \tag{23.12}$$

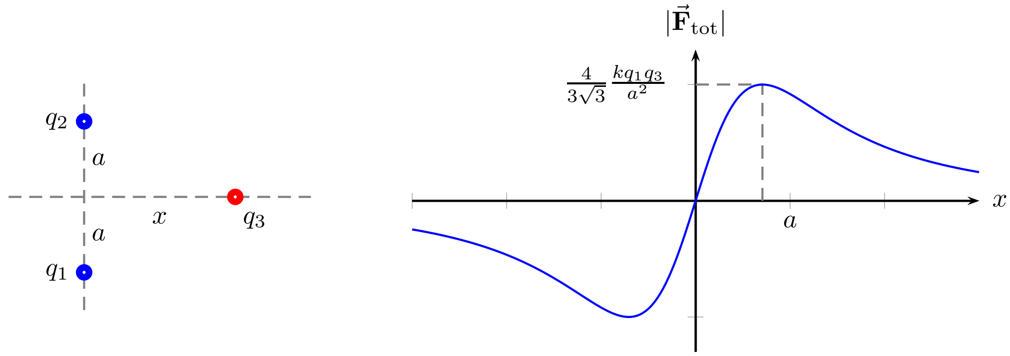

Figure 23.3: Lecture-Example 23.9

- Determine values of x at which the force is a minimum or a maximum? (Ans: $x = \pm\frac{a}{\sqrt{2}}$.)
- What are the minimum and maximum magnitudes of the force?
- You could use $q_1 = q_2 = q_3 = 1.0\,\mu\text{C}$ and $a = 10.0\,\text{cm}$.

Lecture-Example 23.10: (Electroscope)
In Figure 23.4, two tiny conducting balls of identical mass $m = 100.0\,\text{g}$ and identical charge $q = 1.0\,\mu\text{C}$ hang from non-conducting threads of length $L = 1.0\,\text{m}$. The threads make a small angle θ with respect to the vertical.

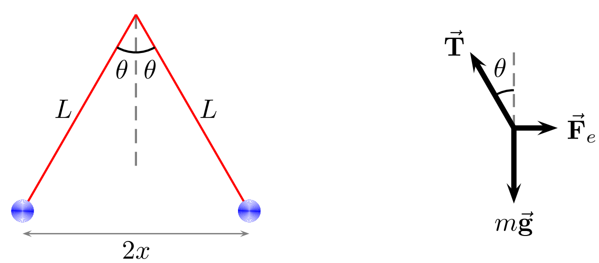

Figure 23.4: Lecture-Example 23.10.

- Show that half of the equilibrium separation, x, between the balls satisfies

$$\frac{x^3}{\sqrt{L^2 - x^2}} = \frac{kq^2}{4mg}. \tag{23.13}$$

This involves solving a cubic equation of the form $y^3 = a^3\sqrt{1 - y^2}$, which has a closed form solution. Here $y = x/L$ and $a^3 = kq^2/4L^2mg$. The series expansion of the solution is

$$y = \begin{cases} a - \dfrac{a^3}{6} - \dfrac{a^5}{72} + \dfrac{5a^7}{1296} + \dfrac{a^9}{384} + \dots, & \text{for} \quad a \ll 1, \\[2ex] 1 - \dfrac{1}{2a^6} + \dfrac{11}{8a^{12}} - \dfrac{85}{16a^{18}} + \dots, & \text{for} \quad a \gg 1, \end{cases} \tag{23.14}$$

where $a < 1$ corresponds to the case when the electrostatic force is larger than the gravitational force and $a > 1$ corresponds to the opposite case. Using these limiting forms qualitatively plot y as a function of a.

- If we discharge one of the spheres, it becomes charge neutral, thus switches off the electrostatic force between the spheres. The spheres will collapse, come in contact, redistribute the charge between them, and again separate due to repulsion. Find the new separation distance.
- This setup can be used to measure q as a function of the angular separation θ, after using $x = L\sin\theta$. Thus, it serves the purpose of a rudimentary electroscope.

Lecture-Example 23.11: Two conducting balls of identical mass $m = 100.0\,\text{g}$ and identical charge $q = 1.0\,\mu\text{C}$ are fixed to the ends of a non-conducting spring of length $L = 1.0\,\text{m}$ (when unstretched) and spring constant k_s. The electrostatic force causes the spring to stretch by a distance x. Show that x satisfies the equation

$$k_s x = \frac{kq^2}{(L+x)^2}. \tag{23.15}$$

23.3 Electric field

Coulomb's law states that an object with non-zero charge exerts a force on another charge with a non-zero charge. In particular, the Coulomb force does not require the two charges to come in contact. How does one charge know to respond to (say the movement of) another charge? That is, how do they communicate? This was not addressed in Coulomb's time and this form of interaction between charges is dubbed action-at-a-distance. Since the time of Faraday, in 1830's, the understanding is that the individual charges are 'immersed' in a 'medium' termed the electric field. The electric field permeates all space and supplies it with an energy and momentum per unit volume. The electric field associates a vector quantity at every point in space at each time. The presence of an individual charge disturbs the electric field continuum, and another charge responds to this disturbance. Further, our understanding is that these disturbances travel at the speed of light as electromagnetic waves. Our current understanding of gravitational interaction is similar, with the curvature tensor taking the role of electric field.

In terms of the electric field the Coulomb force is effectively the same, but for the fact that it is interpreted as a two stage phenomena: the charge q_1 creates an electric field

$$\vec{\mathbf{E}}_1 = \frac{kq_1}{r^2}\hat{\mathbf{r}} \tag{23.16}$$

everywhere in space, which exerts a force

$$\vec{\mathbf{F}}_{21} = q_2\vec{\mathbf{E}}_1 \tag{23.17}$$

on another charge q_2, where $\vec{\mathbf{E}}_1$ is the electric field at the position of charge '2', and $\vec{\mathbf{F}}_{21}$ is read as the force on '2' due to '1'. Conversely, the electric field at a point in space is the force a unit charge would experience if it is placed at the point.

Electric field lines

The electric field associates a vector to every point in space. This information is often represented as electric field lines originating from positive charges and terminating on negative charges. Thus, positive charges are sources of electric field and negative charges are sinks for electric field.

Lecture-Example 23.12: A positively charged sphere, $q = +10.0\,\mu\text{C}$ and $m = 1.00\,\text{g}$, is suspended using a

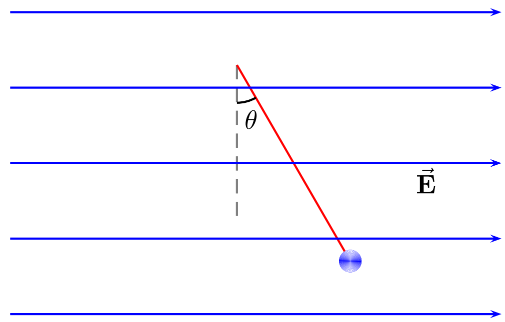

Figure 23.5: Problem 3.

20.0 cm long string in a uniform electric field $E = 1.0 \times 10^3\,\mathrm{N/C}$ as shown in the figure below. Determine the angle θ the string makes with the vertical when the ball is in equilibrium. (Use $g = 10.0\,\mathrm{m/s^2}$.)

Lecture-Example 23.13:
Determine the electric field along the bisector of the line segment connecting two positive charges, $q_1 = q_2 = q$ and distance $2a$.

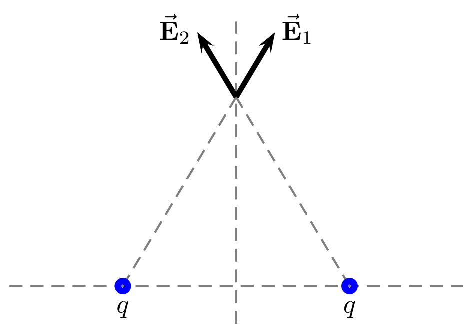

Figure 23.6: Lecture-Example 23.13

- The total electric field at a distance y along the bisector is

$$\vec{\mathbf{E}}_{\text{tot}} = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 = \hat{\mathbf{j}} \frac{2kqy}{(y^2 + a^2)^{\frac{3}{2}}}. \tag{23.18}$$

  See Figure 23.6.

- What is the electric force on charge q_3 at this point.

- Determine the case for $y \gg a$ and $y \ll a$.

Lecture-Example 23.14: (Electric dipole moment)

Two equal and opposite point charges, separated by a distance d, have an electric dipole moment given by

$$\vec{\mathbf{p}} = q\vec{\mathbf{d}}, \tag{23.19}$$

where $\vec{\mathbf{d}}$ points from the negative to the positive charge. Determine the electric field along the bisector of an electric dipole.

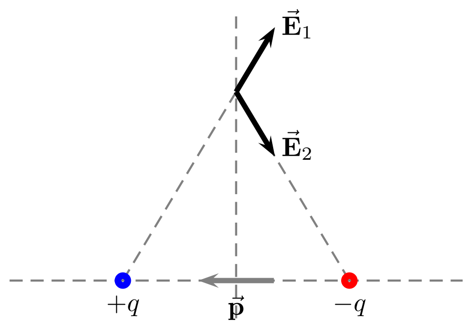

Figure 23.7: Lecture-Example 23.14

- The total electric field at a distance y along the bisector for $d = 2a$ is

$$\vec{\mathbf{E}}_{\text{tot}} = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 = -\frac{k\vec{\mathbf{p}}}{(y^2+a^2)^{\frac{3}{2}}}. \tag{23.20}$$

  See Figure 23.7.

- Unless the atoms are ionized, their interaction with other atoms gets significant contributions from the electric dipole moment. Note that, the electric field due to dipoles has a inverse cube dependence in distance, and thus the corresponding force is much weaker than the Coulomb force.
- The electric field along the line joining the charges is significantly weaker. Thus atoms interacting this way would tend to align in a particular way.
- What is the electric force on charge q_3 at this point.
- Determine the case for $y \gg a$ and $y \ll a$. Observe that for $y \gg a$ it is very weak, but non-zero.

Lecture-Example 23.15: Where is the electric field zero?
See Figure 23.8. Two positive charges q_1 and q_2 are fixed to a line. As a multiple of distance D, at what coordinate on the line is the electric field zero?

- Argue that the electric field goes to zero in the region between the charges. Equate the magnitude of the individual electric fields to deduce

$$x = \frac{D}{\left(1+\sqrt{\frac{q_2}{q_1}}\right)}. \tag{23.21}$$

  For $q_2 > q_1$ we have $0 < x < L/2$. And, for $q_2 < q_1$ we have $0 < L/2 < x < L$. In general the zero-point is closer to the smaller charge.

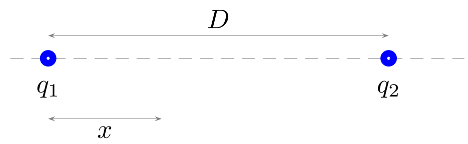

Figure 23.8: Lecture-Example 23.15

- Repeat the above for unlike q_1 and q_2. For this case, argue that the electric field can not go to zero in the region between the charges. Further, using appropriate inequalities involving the charges and the distances, argue that the electric field goes to zero in the region next to the smaller charge.

Lecture-Example 23.16: (Uniformly charged rod)
Show that the electric field due to a uniformly charged rod (of infinite length, charge per unit length λ) at a distance a away from the rod is given by

$$\vec{\mathbf{E}} = \hat{\mathbf{r}}\,\frac{2k\lambda}{a}, \tag{23.22}$$

where $\hat{\mathbf{r}}$ points away from the rod.

Lecture-Example 23.17: (Uniformly charged ring)
Show that the electric field due to a uniformly charged ring of radius r at a distance x away from the ring along the symmetry axis is given by

$$\vec{\mathbf{E}} = \hat{\mathbf{r}}\,\frac{kQx}{(r^2+x^2)^{\frac{3}{2}}}, \tag{23.23}$$

where $\hat{\mathbf{r}}$ points away from the ring along the symmetry axis.

Lecture-Example 23.18: (Uniformly charged plate)
Show that the electric field due to a uniformly charged plate with uniform charge density σ is given by

$$\vec{\mathbf{E}} = \hat{\mathbf{r}}\,\frac{\sigma}{2\varepsilon_0}, \tag{23.24}$$

where $\hat{\mathbf{r}}$ points away from the plate.

- Determine the surface charge density needed to generate an electric field of $100\,\mathrm{N/C}$? (Answer: $1.8\,\mathrm{nC/m^2}$.)

We summarize the electric fields for some relevant geometries here:

$$\text{Point}: \quad \vec{\mathbf{E}} = \hat{\mathbf{r}}\,\frac{kQ}{r^2}, \qquad \hat{\mathbf{r}} \to \text{radially outward (spherically)}, \tag{23.25a}$$

$$\text{Line}: \quad \vec{\mathbf{E}} = \hat{\mathbf{r}}\,\frac{2k\lambda}{r}, \qquad \lambda = \frac{Q}{L}, \qquad \hat{\mathbf{r}} \to \text{radially outward (cylindrically)}, \tag{23.25b}$$

$$\text{(Dielectric) Plane}: \quad \vec{\mathbf{E}} = \hat{\mathbf{r}}\,\frac{\sigma}{2\varepsilon_0} = \hat{\mathbf{r}}\,2\pi k\sigma, \qquad \sigma = \frac{Q}{A}, \qquad \hat{\mathbf{r}} \to \text{normal to the plane}, \tag{23.25c}$$

$$\text{(Conducting) Plane}: \quad \vec{\mathbf{E}} = \hat{\mathbf{r}}\,\frac{\sigma}{\varepsilon_0}, \qquad \sigma = \frac{Q}{A}, \qquad \hat{\mathbf{r}} \to \text{normal to the plane}. \tag{23.25d}$$

23.4 Motion of a charged particle in a uniform electric field

A charged particle experiences a force in an electric field. If the electric force is the only force acting on the charge the corresponding acceleration is

$$\vec{\mathbf{a}} = \frac{q\vec{\mathbf{E}}}{m}. \tag{23.26}$$

Observe that, unlike the case of acceleration in a gravitational field, the acceleration in an electric field is mass dependent. That is, a proton will experience an acceleration 2000 times smaller than that experienced by an electron, because a proton is $\sim$ 2000 times heavier than an electron.

Lecture-Example 23.19:

- Determine the acceleration of a ball of mass $m = 10.0\,\text{g}$ with a charge $q = 1.0\,\mu\text{C}$ in an electric field $E = 1000.0\,\text{N/C}$. (Answer: $0.10\,\text{m/s}^2$.)
  Determine the acceleration of an electron in an electric field $E = 1000.0\,\text{N/C}$. (Answer: $1.8 \times 10^{14}\,\text{m/s}^2$.)
  Determine the acceleration of a proton in an electric field $E = 1000.0\,\text{N/C}$. (Answer: $9.6 \times 10^{10}\,\text{m/s}^2$.)

- Starting from rest, determine the distance travelled by the ball, electron, and the proton, in the presence of this electric field in 1 ns.

- Starting from rest, determine the speed attained by the ball, electron, and the proton, in the presence of this electric field in 1 ns.

Lecture-Example 23.20: (Electric forces are mass dependent)
Recollect from Table 23.1 that the proton and the electron have the same magnitude of charge on them. Further, the proton is 1836 times heavier than the electron.

- A proton and an electron are released from rest in a uniform gravitational field $\vec{\mathbf{g}} = -\hat{\mathbf{z}}\,g$. Find the ratio of the times taken for them to move a distance y.

- A proton and an electron are released from rest in a uniform electric field $\vec{\mathbf{E}} = -\hat{\mathbf{z}}\,E$. Find the ratio of the times taken for them to move a distance y.

Lecture-Example 23.21:
A proton is projected horizontally with an initial speed of $v_i = 1.00 \times 10^5\,\text{m/s}$. It enters a uniform electric field with a magnitude of $E = 100.0\,\text{N/C}$ pointing vertically down. The electric field is confined between plates with a vertical distance $y = 2.0\,\text{cm}$. Determine the horizontal distance x the proton moves before it hits the bottom plate.

- The acceleration experienced by the proton in the y direction due to the electric field is given by

$$a_y = \frac{q}{m}E \sim 9.6 \times 10^9\,\frac{\text{m}}{\text{s}^2}. \tag{23.27}$$

  This is stupendous in comparison to the acceleration due to gravity, $9.8\,\text{m/s}^2$. Thus, we can neglect the gravitational effects all together in this case.

- The kinematics under this constant acceleration are governed by the equations

$$x = v_i t, \tag{23.28a}$$

$$y = \frac{1}{2} a_y t^2. \tag{23.28b}$$

  The second equation here lets us evaluate the time it takes for the proton to fall the distance y as $t = 2.0 \times 10^{-6}\,\mathrm{s}$. This in turn lets us evaluate the horizontal distance x to be $20\,\mathrm{cm}$.

- Repeat the above for an electron. Now we can find $a_y = 1.7 \times 10^{13}\,\mathrm{m/s^2}$, which is about 2000 times larger than that of a proton. The time it takes to hit the bottom plate is $t = 4.9 \times 10^{-8}\,\mathrm{s}$. This leads to $x = 4.9\,\mathrm{mm}$.

- Repeat this for a metal sphere of mass $m = 1.0\,\mathrm{g}$ and charge $q = 10.0\,\mu\mathrm{C}$. Is it reasonable to neglect gravity in this case?

Chapter 24

Gauss's law

24.1 Electric flux

Flux associated with a field $\vec{\mathbf{E}}$ across an infinitesimal area $d\mathbf{A}$ is defined as

$$d\Phi_E = \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}. \tag{24.1}$$

Flux associated with a field $\vec{\mathbf{E}}$ across a surface area S is then given by

$$\Phi_E = \int_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}. \tag{24.2}$$

Electric field lines represent the 'flow' of the electric field, and a quantitative measure of this flow across a surface is the electric flux. It is a measure of the number of electric field lines crossing a surface (presuming a fixed number of lines were originating from sources).

Area in our discussions is a vector. Its magnitude is the area of the surface in context, and its direction is normal to the surface. A surface encloses a volume and the normal to the surface is outward with respect to this volume. For an infinite plane, the ambiguity in the sign of the direction of the normal could be removed if we specify which half it is enclosing.

Lecture-Example 24.1:
The drawing shows an edge-on view of a planar surface of area $2.0\,\text{m}^2$. Given $\theta = 30°$. The uniform electric field $\vec{\mathbf{E}}$ in the drawing has a magnitude of $3.0 \times 10^2\,\text{N/C}$.

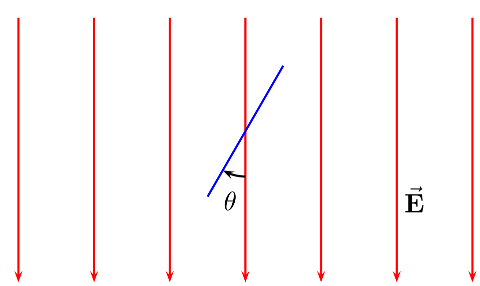

Figure 24.1: Problem 3.

- Calculate the electric flux across the planar surface. Remember that area is a vector normal to the surface. (Answer: $\Phi_E = 3.0 \times 10^2\,\text{Nm}^2/\text{C}$.)

Lecture-Example 24.2:
Consider a sheet of paper folded and kept in a uniform electric field $\vec{\mathbf{E}} = E_0\hat{\mathbf{x}}$, with $E_0 = 100.0\,\mathrm{N/C}$. The vertical side of the area along $\hat{\mathbf{y}}$ is 10.0 cm in length and it is 10.0 cm deep in the $\hat{\mathbf{z}}$ direction. The inclined side has the same height in the $\hat{\mathbf{y}}$ and makes 60.0° with respect to the vertical. Calculate the flux across surface S_1 and S_2.

- The flux across surface S_1 is given by

$$\Phi_E^{S_1} = \int_{S_1} E_0\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}dA = \int_{S_1} E_0 dA = E_0 A_1, \tag{24.3}$$

  where we used $d\vec{\mathbf{A}} = \vec{\mathbf{x}}\,dA$.

- The flux across surface S_2 is given by

$$\Phi_E^{S_2} = \int_{S_2} E_0\hat{\mathbf{x}}\cdot\hat{\mathbf{n}}dA = \int_{S_2} E_0\cos\theta dA = E_0 A_2\cos\theta = E_0 A_1, \tag{24.4}$$

  where we used $d\vec{\mathbf{A}} = \vec{\mathbf{n}}\,dA$, and $\vec{\mathbf{x}}\cdot\vec{\mathbf{n}} = \cos\theta$, and $A_1 = A_2\cos\theta$.

Lecture-Example 24.3:
Consider a uniform electric field $\vec{\mathbf{E}} = E_0\hat{\mathbf{x}}$. A cube, of edge length $L = 10.0\,\mathrm{cm}$, is placed in this electric field with one of the faces perpendicular to the field. Find the electric flux across each of the faces of the cube. Find the total flux across the surface of the cube.

Lecture-Example 24.4:
Consider a region of uniform electric field

$$\vec{\mathbf{E}} = (1.0\,\hat{\mathbf{i}} + 2.0\,\hat{\mathbf{j}})\times 10^3\,\frac{\mathrm{N}}{\mathrm{C}}. \tag{24.5}$$

Calculate the electric flux through a rectangular plane 0.40 m wide and 0.20 m long if the plane is parallel to the yz plane.

Lecture-Example 24.5:
Flux across a sphere enclosing a point charge at the center.

- Using

$$\vec{\mathbf{E}} = \frac{kQ}{r^2}\,\hat{\mathbf{r}} \qquad \text{and} \qquad d\vec{\mathbf{A}} = \hat{\mathbf{r}}\,dA \tag{24.6}$$

  the flux is given by

$$\Phi = \oint_S \vec{\mathbf{E}}\cdot d\mathbf{A} = \frac{kQ}{r^2}\oint_S dA = \frac{kQ}{r^2}4\pi r^2 = \frac{Q}{\varepsilon_0}. \tag{24.7}$$

24.2 Gauss's law

Gauss's law states that the electric flux across a closed surface is completely determined by the total charge enclosed inside the surface,

$$\Phi_E = \oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{\text{en}}}{\varepsilon_0}, \tag{24.8}$$

where Q_{en} is the total charge enclosed inside the closed surface S.

Lecture-Example 24.6: (Point charge)
Using the symmetry of a point charge, and presuming the electric field to be radial and isotropic, derive Coulomb's law using Gauss's law,

$$\vec{\mathbf{E}} = \frac{kQ}{r^2}\,\hat{\mathbf{r}}. \tag{24.9}$$

Lecture-Example 24.7: (Charged spherical shell)
Determine the electric field inside and outside a uniformly charged spherical shell to be

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{kQ}{r^2}\,\hat{\mathbf{r}}, & R < r, \\ 0, & r < R. \end{cases} \tag{24.10}$$

- This suggests that we can not infer about the charge distribution of a sphere based on the measurement of electric field outside the sphere. For example, what can we say about the charge distribution of proton, that is, is it a uniformly charged solid or a shell?
- By analogy, we can conclude that the acceleration due to gravity inside a spherical shell with uniform mass density on the surface will be zero.

Lecture-Example 24.8: (Perfect Conductor of arbitrary shape)
Prove that the electric field inside a conductor of arbitrary shape is exactly zero.

- Inside a conductor is the safest place during lightning.

Lecture-Example 24.9: (Structure of an atom)
The electric field inside and outside a uniformly charged solid sphere of radius R and charge Q is given by

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{kQ}{r^2}\,\hat{\mathbf{r}}, & R < r, \\ \dfrac{kQ}{R^3} r\,\hat{\mathbf{r}}, & r < R. \end{cases} \tag{24.11}$$

- Plot the magnitude of the electric field as a function of r.
- Discuss how this contributed to the Rutherford's model for the structure of atom.

Lecture-Example 24.10:
Consider a perfectly conducting sphere of radius $R = 3.0\,\text{cm}$ with charge $Q = 1.0\,\mu\text{C}$. Determine the electric flux through the surface of a (Gaussian) sphere of radius 2.0 cm, concentric with respect to the conducting sphere.

Lecture-Example 24.11: (Uniformly charged line)
Using Gauss's law determine the electric field near a uniformly charged line (of infinite extent) with charge per unit length λ to be

$$\vec{\mathbf{E}} = \frac{2k\lambda}{r}\,\hat{\mathbf{r}}, \tag{24.12}$$

where $\hat{\mathbf{r}}$ is the unit vector perpendicular to the line charge.

Lecture-Example 24.12: (Conducting versus non-conducting plate)
Using Gauss's law show that the electric field near a uniformly charged plate is constant and given by

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{\sigma}{2\varepsilon_0}, & \text{non-conducting (dielectric) plate,} \\ \dfrac{\sigma}{\varepsilon_0}, & \text{conducting plate,} \end{cases} \tag{24.13}$$

where σ is the charge per unit area on the plate.

24.3 Homework problems

Homework-Problem 24.6: A charge of $105\,\mu\text{C}$ is at the center of a cube of edge 75.0 cm. No other charges are nearby.

1. Find the flux through each face of the cube.
2. Find the flux through the whole surface of the cube.
3. Would your answers to parts (a) or (b) change if the charge were not at the center?

Hints:

- Solve for second part using Gauss's law, $\Phi_\text{E} = \frac{Q_\text{encl}}{\epsilon_0}$, to find flux through the whole surface.
- Notice that flux through one face is one-sixth of the flux through the whole surface

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Homework-Problem 24.8: The charge per unit length on a long, straight filament is $-88.5\,\mu\text{C/m}$.

1. Find the electric field 10.0 cm from the filament, where distances are measured perpendicular to the length of the filament. (Take radially inward toward the filament as the positive direction.)
2. Find the electric field 21.0 cm from the filament, where distances are measured perpendicular to the length of the filament.

3. Find the electric field 110 cm from the filament, where distances are measured perpendicular to the length of the filament.

Hints: Use

$$\vec{\mathbf{E}} = \hat{\mathbf{r}}\frac{2k_e\lambda}{r} = \hat{\mathbf{r}}\frac{\lambda}{2\pi\varepsilon_0 r}. \tag{24.14}$$

Homework-Problem 24.9: A large, flat, horizontal sheet of charge has a charge per unit area of $2.30\,\mu\text{C/m}^2$. Find the electric field just above the middle of the sheet.

Hints: Use

$$\vec{\mathbf{E}} = \hat{\mathbf{n}}\frac{\sigma}{2\varepsilon_0}. \tag{24.15}$$

Homework-Problem 24.10: Consider a thin, spherical shell of radius 15.0 cm with a total charge of $32.2\,\mu$C distributed uniformly on its surface.

1. Find the electric field 10.0 cm from the center of the charge distribution.
2. Find the electric field 22.0 cm from the center of the charge distribution.

Hints: Use

$$\vec{\mathbf{E}} = \hat{\mathbf{r}}\frac{k_e Q}{r^2}. \tag{24.16}$$

Homework-Problem 24.12: Two identical conducting spheres each having a radius of 0.500 cm are connected by a light 2.10 m long conducting wire. A charge of $56.0\,\mu$C is placed on one of the conductors. Assume the surface distribution of charge on each sphere is uniform. Determine the tension in the wire.

Hints: The charge will get equally distributed on the two conducting spheres. The repulsive electrostatic force between two spheres is balanced by the tension in the wire. (*Conducting sphere with charge on its surface behaves like a point charge.*)

Chapter 25

Electric potential energy and electric potential

25.1 Work done by the electric force

The electric force on a charge q in an electric field $\vec{\mathbf{E}}$ is given by

$$\vec{\mathbf{F}} = q\vec{\mathbf{E}}. \tag{25.1}$$

The work done by the electric force on charge q is given by

$$W = \int_a^b \vec{\mathbf{F}} \cdot d\vec{\mathbf{l}} = q \int_a^b \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}, \tag{25.2}$$

where the integral is evaluated along a path connecting the position points $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

Lecture-Example 25.1: Consider a region of uniform electric field $\vec{\mathbf{E}} = -E\,\hat{\mathbf{j}}$ of magnitude $E = 1.0 \times 10^3\,\text{N/C}$ and direction vertically down. Determine the work done by the electric force when a charged sphere with charge $q = 10.0\,\mu\text{C}$ is moved along a path. Let the vertical distance between points '1' to '2' be $h = 10.0\,\text{cm}$.

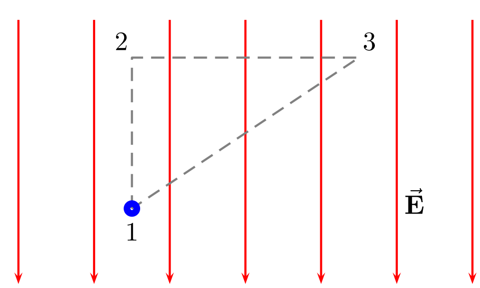

Figure 25.1: Lecture-Example 25.1

- The work done by the electric force when the particle moves along the path connecting points '1' to '2', '2' to '3', and '3' to '1', are

$$W_{1\to 2} = -qEh, \tag{25.3a}$$
$$W_{2\to 3} = 0, \tag{25.3b}$$
$$W_{3\to 1} = qEh. \tag{25.3c}$$

- Further, the total work done by the electric force for the closed loop $1 \to 2 \to 3 \to 1$ is

$$W_{1\to 2\to 3\to 1} = 0. \tag{25.4}$$

- Note that the work done is zero for a path element that is perpendicular to the electric field. An arbitrary path can be broken down into infinitely small vertical and horizontal displacements. Thus, for the case of uniform electric field we can show that the work done is independent of the path and only depends on the initial and final points.

25.2 Electric potential energy

The work done by the electric force is zero for a closed path (in the absence of time varying magnetic fields),

$$\oint q\vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = 0. \tag{25.5}$$

As a corollary, the work done by the electric force is completely determined by the initial and final points of the path traversed. This is the statement of the electric force being a conservative force. For a conservative force it is convenient to define an associated potential energy, in the statement of work-energy theorem. Thus we define the electric potential energy as

$$\Delta U = U_f - U_i = -W_{i\to f} = -\int_i^f q\vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}. \tag{25.6}$$

Lecture-Example 25.2: (Point charges)
A positive charge q_2 is moved in the vicinity of a another positive charge q_1. Determine the work done by the electric force when the charge q_2 is moved along a path.

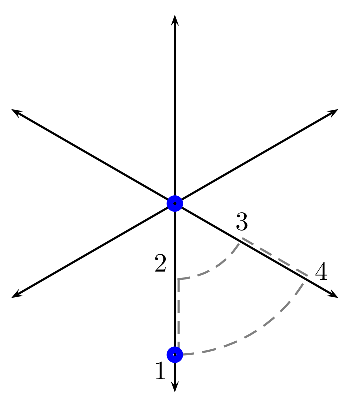

Figure 25.2: Lecture-Example 25.2

- The work done by the electric force on the charge q_2 when it is moved along the path '1' to '2' is

$$W_{1\to 2} = \int_1^2 q_2\vec{\mathbf{E}}_1 \cdot d\vec{\mathbf{l}} = \int_1^2 \frac{kq_1q_2}{r^2}\,\hat{\mathbf{r}} \cdot d\vec{\mathbf{l}} = \int_1^2 \frac{kq_1q_2}{r^2}\,dr = -\frac{kq_1q_2}{r}\bigg|_1^2, \tag{25.7}$$

where we used $d\vec{\mathbf{l}} = -\hat{\mathbf{r}}dl = \hat{\mathbf{r}}dr$. Similarly we can evaluate $W_{2\to 3}$ $W_{3\to 4}$, and $W_{4\to 1}$.

- If we choose the initial point to be at infinity, the work done by the electric force while charge q_2 is moved from infinity to a distance r away from charge q_1 is

$$U = \frac{kq_1 q_2}{r}. \tag{25.8}$$

- Plot the electric potential energy U between two positive charges q_1 and q_2 as a function of r. Next, plot the electric potential energy U between two unlike charges q_1 and q_2 as a function of r. Interpret these plots as a statement of the fact that force is the manifestation of the system trying to minimize its energy.
- Equipotential surfaces are surfaces perpendicular to the electric field. The work done by the electric force is zero while moving on equipotential surfaces.

Lecture-Example 25.3: (Energy required to assemble a set of charges)
Show that the energy required to assemble three positive charges q_1, q_2, and q_3, at relative distances r_{12}, r_{23}, and r_{31}, is

$$U = \frac{kq_1 q_2}{r_{12}} + \frac{kq_2 q_3}{r_{23}} + \frac{kq_3 q_1}{r_{31}}. \tag{25.9}$$

- Show that the total energy required to assemble three identical positive charges q at the corners of an equilateral triangle of side L is

$$U = 3\frac{kq^2}{L}. \tag{25.10}$$

- Show that the total energy required to assemble four identical positive charges q at the corners of a square of side L is

$$U = (4 + \sqrt{2})\frac{kq^2}{L}. \tag{25.11}$$

Lecture-Example 25.4:
A sphere with mass $m_2 = 10\,\text{g}$ and charge $q_2 = 1.0\,\mu\text{C}$ is fired directly toward another sphere of charge $q_1 = 10.0\,\mu\text{C}$ (which is pinned down to avoid its motion). If the initial velocity of charge q_2 is $v_i = 10.0\,\text{m/s}$ when it is $r_i = 30\,\text{cm}$ away from charge q_1, at what distance away from the charge q_1 does it come to rest?

- Using conservation of energy we have

$$\frac{kq_1 q_2}{r_i} + \frac{1}{2}m_2 v_i^2 = \frac{kq_1 q_2}{r_f} + \frac{1}{2}m_2 v_f^2. \tag{25.12}$$

Answer: $r_f = 11\,\text{cm}$.

Lecture-Example 25.5:
Two oppositely charged, parallel plates are placed $d = 8.0\,\text{cm}$ apart to produce an electric field of strength $E = 1.0 \times 10^3\,\text{N/C}$ between the plates. A sphere of mass $m = 10.0\,\text{g}$ and charge $q = 10.0\,\mu\text{C}$ is projected from one surface directly toward the second. What is the initial speed of the sphere if it comes to rest just at the second surface?

- Using conservation of energy we have

$$\frac{1}{2}mv^2 = qEd. \tag{25.13}$$

Answer: $v = 0.4\,\text{m/s}$.

25.3 Electric potential

Electric potential energy per unit charge is defined as the electric potential. It is measured in units of Volt=Joule/Coulomb. Thus,

$$\Delta U = q\Delta V. \tag{25.14}$$

Thus,

$$\Delta V = -\int_i^f \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}. \tag{25.15}$$

For a point charge, after choosing the electric potential to be zero at infinity, we have

$$V = \frac{kq}{r}. \tag{25.16}$$

For uniform electric field created by oppositely charged parallel plates, after choosing the electric potential to be zero at the negative plate, we have

$$V = Ed, \tag{25.17}$$

d being the distance from the negative plate.

Lecture-Example 25.6:
Two electrons and two protons are placed at the corners of a square of side 5 cm, such that the electrons are at diagonally opposite corners.

- What is the electric potential at the center of square?
- What is the electric potential at the midpoint of either one of the sides of the square?
- How much potential energy is required to move another proton from infinity to the center of the square?
- How much additional potential energy is required to move the proton from the center of the square to one of the midpoint of either one of the sides of the square?

Lecture-Example 25.7:
The two charges in Figure 25.3 are separated by a distance a. Let $a = 5.00\,\text{cm}$, $q = 5.00\,\text{nC}$, $Q = 1.00\,\mu\text{C}$.

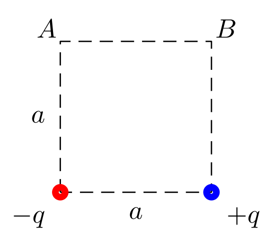

Figure 25.3: Lecture-Example 25.7

- Find the electric potential at point A, choosing the potential at infinity to be zero.
  Answer: $V_A = -\frac{kq}{a}\left(1 - \frac{1}{\sqrt{2}}\right) = -263\,\text{V}$.

- Find the electric potential at point B, choosing the potential at infinity to be zero.
  Answer: $V_B = +\frac{kq}{a}\left(1 - \frac{1}{\sqrt{2}}\right) = +263\,\text{V}$.

- Determine the potential difference between points A and B.
  Answer: $V_B - V_A = +2\frac{kq}{a}\left(1 - \frac{1}{\sqrt{2}}\right) = +527\,\text{V}$.

- Determine the change in electric potential energy when a third charge of Q is moved from point A to B.
  Answer: $\Delta U = +2\frac{kqQ}{a}\left(1 - \frac{1}{\sqrt{2}}\right) = +527\,\mu\text{J}$.

Lecture-Example 25.8:
Charges of $-q$ and $+2q$ are fixed in place, with a distance of $a = 2.0\,\text{m}$ between them. See Fig. 25.4. A dashed line is drawn through the negative charge, perpendicular to the line between the charges.

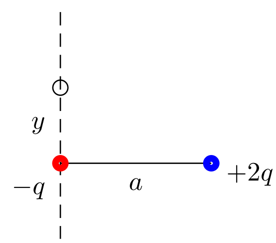

Figure 25.4: Lecture-Example 25.8

- On the dashed line, at a distance y from the negative charge, there is at least one spot where the total potential is zero. Find y. (Answer: $y = \pm a/\sqrt{3}$.)

- On the line connecting the charges, between the two charges, find the spot where the total potential is zero. (Answer: Distance $a/3$ to the right of $-q$ charge.) On the line connecting the charges, to the left of the smaller charge, find the spot where the total potential is zero. (Answer: Distance a to the left of $-q$ charge.)

- On the line connecting the charges, to the right of the larger charge, show that there is no spot where the total potential is zero. In general for $\alpha = q_2/q_1 < 0$, remembering that the potential involves the magnitude of the distance, the two solutions on the line connecting the charges are contained as solutions to the quadratic equation,
$$(\alpha^2 - 1)z^2 + 2az - a^2 = 0, \tag{25.18}$$
which has solutions
$$z = \frac{a}{1+\alpha} \quad \text{and} \quad z = \frac{a}{1-\alpha}. \tag{25.19}$$

- For like charges, $\alpha = q_2/q_1 > 0$, there is no spot with zero potential other than infinity, because two positive numbers can not add to give zero.

- Determine the equation of the equipotential surface of zero potential. Discuss the shape of this surface with respect to α.

25.4 Gradient

The gradient $\vec{\boldsymbol{\nabla}}$ of a function $f(x, y, z)$ is defined as

$$\vec{\boldsymbol{\nabla}} f(x,y,z) = \hat{\mathbf{i}}\,\frac{\partial f}{\partial x} + \hat{\mathbf{j}}\,\frac{\partial f}{\partial y} + \hat{\mathbf{k}}\,\frac{\partial f}{\partial z}. \tag{25.20}$$

25.5 Force as gradient of energy

For conservative forces the integral in Eq. (25.6) can be inverted to yield

$$\vec{\mathbf{F}} = -\vec{\boldsymbol{\nabla}} U. \tag{25.21}$$

This states that the force is the manifestation of the system trying to minimize its energy. Similarly, Eq. (25.15) can be inverted to yield

$$\vec{\mathbf{E}} = -\vec{\boldsymbol{\nabla}} V, \tag{25.22}$$

which determines the electric field as the gradient (derivative) of the electric potential.

Lecture-Example 25.9: (Interaction energy for point charges)
Plot the electric potential energy of two charges as a function of separation distance between the charges. Choose potential energy to be zero for infinite separation distance. Consider both like and unlike charges. Interpret force as the negative derivative of potential energy. Repeat this for electric potential, choosing it to be zero at infinity.

Lecture-Example 25.10: (Parallel plates)
Consider uniformly charged parallel plates containing opposite charges. Plot the electric potential as a function of the distance from the negative plate. Choose the electric potential to be zero at the negative plate.

Lecture-Example 25.11: (Two point charges)
Consider two positive charges q, both a distance a from the origin such that their separation distance is $2a$. Determine the electric potential on the x axis to be

$$V(x) = \frac{2kq}{\sqrt{x^2 + a^2}}. \tag{25.23}$$

Using the fact that the electric field is the negative gradient of the electric potential, calculate the x-component of the electric field vector on the x axis to be

$$E_x = -\frac{\partial V}{\partial x} = \frac{2kqx}{(x^2 + a^2)^{\frac{3}{2}}}. \tag{25.24}$$

- Discuss the limits $x \ll a$ and $a \ll x$.

Lecture-Example 25.12: (Ring)

Consider a uniformly charged ring of radius r with total charge Q placed on the yz plane such that the origin is the center of the ring. Determine the electric potential on the x axis to be

$$V(x) = \frac{kQ}{\sqrt{x^2 + r^2}}. \tag{25.25}$$

Using the fact that the electric field is the negative gradient of the electric potential, calculate the x-component of the electric field vector on the x axis to be

$$E_x = -\frac{\partial V}{\partial x} = \frac{kQx}{(x^2 + r^2)^{\frac{3}{2}}}. \tag{25.26}$$

- Discuss the limits $r \ll a$ and $a \ll r$.

Lecture-Example 25.13: (Disc)
Consider a uniformly charged disc of radius R with charge per unit area σ placed on the yz plane such that the origin is the center of the disc. Determine the electric potential on the x axis to be

$$V(x) = -\frac{\sigma}{2\varepsilon_0}\left[x - \sqrt{x^2 + R^2}\right]. \tag{25.27}$$

Using the fact that the electric field is the negative gradient of the electric potential, calculate the x-component of the electric field vector on the x axis to be

$$E_x = -\frac{\partial V}{\partial x} = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{x}{\sqrt{x^2 + R^2}}\right]. \tag{25.28}$$

- Show that this leads to the potential and electric field of a point charge in the limit $R \ll x$.
- Analyze the limit $x \ll R$. Plot the electric potential as a function of a x for this case.

25.6 Electric potential inside a perfect conductor

The electric field is zero inside a perfect conductor, (otherwise the charges will experience a force,) which implies that the electric potential is a constant inside the conductor.

Lecture-Example 25.14:
Determine the electric potential inside and outside a perfectly conducting charged sphere of radius R. Plot this.

Lecture-Example 25.15: (Fork in a microwave)
To illustrate why pointed metals spark inside a microwave, let us consider two conducting spheres of radius R_1 and R_2, connected by a conducting thread, but placed significantly away from each other.

- Using the fact that the electric potential is the same at the surface of the two spheres,

  $$V_1 = V_2, \tag{25.29}$$

  show that the ratio of the charges on the two spheres is

  $$\frac{Q_1}{Q_2} = \frac{R_1}{R_2}. \tag{25.30}$$

  Thus, the charge is proportional to the radius.

- Show that the ratio of the electric fields is

$$\frac{E_1}{E_2} = \frac{R_2}{R_1}. \tag{25.31}$$

  Thus, the electric field is inversely proportional to the radius. This implies the smaller sphere will have a larger electric field near its surface. If the electric field is large enough to breakdown air, we see a spark.

Chapter 26

Capacitance

We have learned that positive charges tend to move from a point of higher electrical potential to a point of lower electrical potential, and negative charges tend to do the opposite. This basic idea is at the heart of electrical circuits, which involves flow of electric charges. A traditional battery is a device that provides a (constant) potential difference, by moving charges against their natural tendency. The three basic electrical components will will discuss are: capacitor, resistor, and inductor.

26.1 Capacitor

It is often desirable to have a reservoir of charge. A capacitor is an electrical component that can store charge. It should be distinguished from a traditional battery whose purpose is to provide a potential difference. To this end, consider two conducting objects, of arbitrary shape, with equal and opposite charge Q on them. The charged objects create a potential difference, which is proportional to the charge,

$$V = \frac{Q}{C}, \tag{26.1}$$

where the voltage V is the potential difference between the objects, effectively same as temporarily choosing the negative plate to be at zero potential. Comparing Eq. (26.1) to the potential due to a point charge, we learn that the capacitance has the dimensions given by

$$[C] = [\varepsilon_0]L. \tag{26.2}$$

Capacitance is measured in units of Farad, a derived unit. Equation (26.2) suggests that the capacitance is completely determined by the geometry of the two conducting objects, and the permittivity of the medium.

Lecture-Example 26.1: (Parallel plate capacitor)
The electric field between parallel conducting plates, with uniform surface charge density $\sigma = Q/A$, is given by Eq. (23.25d). (Using Gauss's law the electric field outside the plates is zero.) Determine the potential difference between the plates using Eq. (25.15). Comparing this with Eq. (26.1) we identify the capacitance for a parallel plate capacitor to be

$$C = \frac{\varepsilon_0 A}{d}, \tag{26.3}$$

where A is the area of the plates and d is the separation distance of the plates. Verify that the positively charged plate is at higher electric potential.

Lecture-Example 26.2: (Cylindrical capacitor)

The electric field between coaxial conducting cylinders, with uniform line charge density $\lambda = Q/L$, is given by Eq. (23.25b). (Using Gauss's law the electric field outside the outer cylinder and inside the inner cylinder is zero.) Determine the potential difference between the cylinders using Eq. (25.15). Comparing this with Eq. (26.1) we identify the capacitance for a parallel plate capacitor to be

$$C = \frac{2\pi\varepsilon_0 L}{\ln \frac{b}{a}}, \tag{26.4}$$

where L is the length of the cylinder, and a and b are the radius of the inner and outer cylinders respectively. Verify that the positively charged cylinder, irrespective of it being inside or outside, is at higher electric potential.

Lecture-Example 26.3: (Spherical capacitor)
The electric field between concentric conducting spheres, with charge Q uniformly distributed on the spheres, is given by Eq. (23.25a). (Using Gauss's law the electric field outside the outer sphere and inside the inner sphere is zero.) Determine the potential difference between the spheres using Eq. (25.15). Comparing this with Eq. (26.1) we identify the capacitance for a parallel plate capacitor to be

$$C = \frac{4\pi\varepsilon_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}, \tag{26.5}$$

where a and b are the radius of the inner and outer spheres respectively. Verify that the positively charged sphere, irrespective of it being inside or outside, is at higher electric potential.

Lecture-Example 26.4: (A rudimentary capacitor)
Cut out two strips of aluminum foil, $A = 1\,\text{cm} \times 1\,\text{m} = 10^{-2}\,\text{m}^2$. Place a sheet of paper in between the strips and roll the sheets. Estimate the thickness of paper to be $d = 100\,\mu\text{m}$. The medium between the plates is paper, which has a permittivity of $\varepsilon \sim 3.9\,\varepsilon_0$. Estimate the capacitance of this construction. (Answer: $C \sim 1\,\text{nF}$.)

- Traditional capacitors used in electrical circuits range between picofarad (pF) and microfarad (μF). Parasitic capacitance, the unavoidable stray capacitance, is typically about 0.1 pF. More recently, capacitance greater than kilofarad (kF) have been feasible, and are called supercapacitors.

26.2 Energy stored in a capacitor

The energy stored in a capacitor can be interpreted as the amount of work done to place the charges on to the conducting plates. Starting from two neutral plates we can achieve this by moving a small charge dq in one step, and repeating this until we collect a total charge of Q. In each step the work required to achieve this is

$$dU = V\,dq = \frac{1}{C} q dq. \tag{26.6}$$

Notice that the work required in successive steps increases linearly. The total work is the integral of the above expression, which is also the area of the triangle in the V-q plot. The energy stored in a capacitor is thus determined to be

$$U = \frac{Q^2}{2C} = \frac{1}{2} C V^2. \tag{26.7}$$

The energy in a capacitor is stored in the form of electric field. For the case of parallel plates, this is explicitly obtained by rewriting the expression for energy in terms of the electric field E,

$$u_E = \frac{U}{Ad} = \frac{1}{2}\varepsilon_0 E^2. \tag{26.8}$$

Lecture-Example 26.5:
The breakdown field strength of paper is about ten times that of air, $E_c \sim 10^7\,\mathrm{V/m}$. Thus, determine the maximum energy that can be stored in the rudimentary capacitor of Lecture-Example 26.4. (Answer: $\sim 1\,\mathrm{mJ}$.)

26.3 Capacitors in series and parallel

A capacitor when connected to a battery collects equal and opposite charges on its plates. The amount of charge Q it collects is decided by the capacitance C and potential difference V across the plates,

$$Q = CV. \tag{26.9}$$

Lecture-Example 26.6:
A capacitor of capacitance $C = 10\,\mu\mathrm{F}$ is connected across a 10.0 V battery. Determine the charge accumulated on the plates of the capacitor. (Answer: $100\,\mu\mathrm{C}$.)

Capacitors in series

Consider two capacitors in series as described in Figure 26.1. Since the potential difference across the battery is distributed across the two capacitors we deduce that

$$V = V_1 + V_2. \tag{26.10}$$

The charges on each of the capacitors will be identical,

$$Q_1 = Q_2, \tag{26.11}$$

because by construction the part of circuit between the two capacitors is isolated. An equivalent capacitor C_{eq} shown on the right side in Figure 26.1 is defined as a capacitor that will collect the same amount of charge from the battery. Thus, using $V_1 = Q_1/C_1$, $V_2 = Q_2/C_2$, and $V = Q_{\mathrm{eq}}/C_{\mathrm{eq}}$, in Eq. (26.10), we learn that

$$\frac{1}{C_{\mathrm{eq}}} = \frac{1}{C_1} + \frac{1}{C_2}. \tag{26.12}$$

We can further deduce that

$$\frac{V_1}{V_2} = \frac{C_2}{C_1}, \tag{26.13}$$

which turns out to be handy in the analysis of more complicated configurations.

Lecture-Example 26.7: (Capacitors in series)
A potential difference $V = 10.0\,\mathrm{V}$ is applied across a capacitor arrangement with two capacitances connected in series, $C_1 = 10.0\,\mu\mathrm{F}$ and $C_2 = 20.0\,\mu\mathrm{F}$.

- Find the equivalent capacitance. (Answer: $C_{\mathrm{eq}} = 6.67\,\mu\mathrm{F}$.)

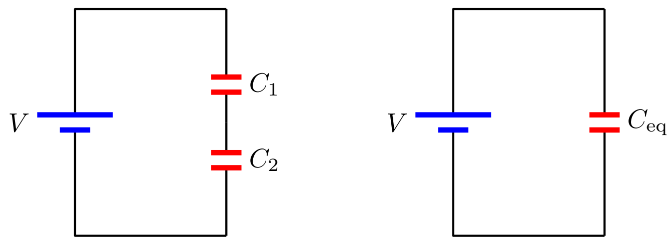

Figure 26.1: Capacitors in series.

- Find the charges Q_1 and Q_2 on each of the capacitors. (Answer: $Q_1 = Q_2 = 66.7\,\mu$C.)
- Find the voltages V_1 and V_2 across each of the capacitors. (Answer: $V_1 = 6.67\,$V and $V_2 = 3.33\,$V.)
- Find the potential energies U_1 and U_2 stored inside each of the capacitors. (Answer: $U_1 = 222\,\mu$J and $U_2 = 111\,\mu$J.)

Capacitors in parallel

Consider two capacitors in parallel as described in Figure 26.2. The potential difference across each capacitor is identical,

$$V = V_1 = V_2. \tag{26.14}$$

The total charge Q that is pulled out of the battery distributes on the two capacitors,

$$Q = Q_1 + Q_2. \tag{26.15}$$

An equivalent capacitor $C_{\rm eq}$ shown on the right side in Figure 26.2 is defined as a capacitor that will collect the same amount of charge from the battery. Thus, using $Q_1 = V_1 C_1$, $Q_2 = V_2 C_2$, and $Q = V C_{\rm eq}$, in Eq. (26.15), we learn that

$$C_{\rm eq} = C_1 + C_2. \tag{26.16}$$

We can further deduce that

$$\frac{Q_1}{Q_2} = \frac{C_1}{C_2}, \tag{26.17}$$

which turns out to be handy in the analysis of more complicated configurations.

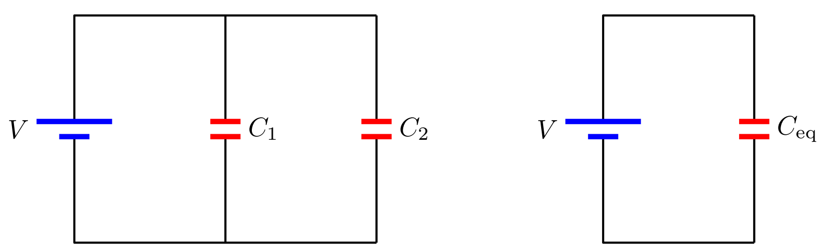

Figure 26.2: Capacitors in parallel.

Lecture-Example 26.8: (Capacitors in parallel)
A potential difference $V = 10.0\,$V is applied across a capacitor arrangement with two capacitances connected in parallel, $C_1 = 10.0\,\mu$F and $C_2 = 20.0\,\mu$F.

- Find the equivalent capacitance. (Answer: $C_{\text{eq}} = 30.0\,\mu\text{F}$.)
- Find the voltages V_1 and V_2 across each of the capacitors. (Answer: $V_1 = V_2 = 10.0\,\text{V}$.)
- Find the charges Q_1 and Q_2 on each of the capacitors. (Answer: $Q_1 = 0.100\,\text{mC}$ and $Q_2 = 0.200\,\text{mC}$.)
- Find the potential energies U_1 and U_2 stored inside each of the capacitors. (Answer: $U_1 = 0.500\,\text{mJ}$ and $U_2 = 1.00\,\text{mJ}$.)

Lecture-Example 26.9:
Consider the circuit in Figure 26.3. Let $V = 10.0\,\text{V}$, $C_1 = 10.0\,\mu\text{F}$, $C_2 = 20.0\,\mu\text{F}$, and $C_3 = 30.0\,\mu\text{F}$.

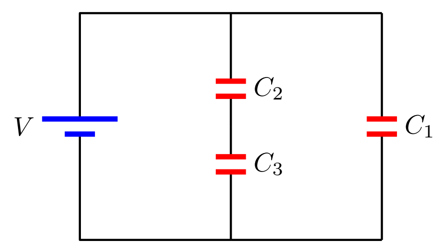

Figure 26.3: Capacitor circuit.

- Determine the equivalent capacitance of the complete circuit. (Answer: $C_{\text{eq}} = 22.0\,\mu\text{F}$.)
- Determine the charge on each capacitor. (Answer: $Q_1 = 100\,\mu\text{C}$, $Q_2 = Q_3 = 120\,\mu\text{C}$.)
- Determine the voltage across each capacitor. (Answer: $V_1 = 10.0\,\text{V}$, $V_2 = 6.0\,\text{V}$, $V_3 = 4.0\,\text{V}$.)
- Determine the energy stored in each capacitor. (Answer: $U_1 = 500\,\mu\text{J}$, $U_2 = 360\,\mu\text{J}$, $U_3 = 240\,\mu\text{J}$.)

Lecture-Example 26.10:
Consider the circuit in Figure 26.4. Let $V = 10.0\,\text{V}$, $C_1 = 10.0\,\mu\text{F}$, $C_2 = 20.0\,\mu\text{F}$, and $C_3 = 30.0\,\mu\text{F}$. Determine the charges on each capacitor and voltages across each capacitor.

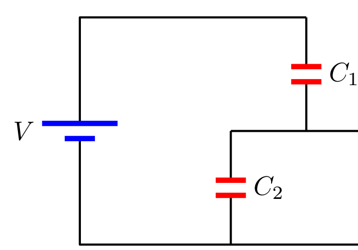

Figure 26.4: Capacitor circuit.

- Determine the equivalent capacitance of the complete circuit. (Answer: $C_{\text{eq}} = 8.33\,\mu\text{F}$.)
- Determine the charge on each capacitor. (Answer: $Q_1 = 83.3\,\mu\text{C}$, $Q_2 = 33.4\,\mu\text{C}$, $Q_3 = 50.1\,\mu\text{C}$.)

- Determine the voltage across each capacitor. (Answer: $V_1 = 8.33\,\text{V}$, $V_2 = V_3 = 1.67\,\text{V}$.)
- Determine the energy stored in each capacitor. (Answer: $U_1 = 347\,\mu\text{J}$, $U_2 = 27.9\,\mu\text{J}$, $U_3 = 41.8\,\mu\text{J}$.)

26.4 Electric dipole in a uniform electric field

An atom is charge neutral. But, an atom's center of positive charge need not coincide with the atom's center of negative charge. Thus, an atom can be effectively described as two equal and opposite charges separated by a distance d. The electric dipole moment $\vec{\mathbf{p}}$ of such a configuration is given by

$$\vec{\mathbf{p}} = q\vec{\mathbf{d}}, \tag{26.18}$$

where the vector $\vec{\mathbf{d}}$ points from the negative charge to the positive charge. The total force on an electric dipole in a uniform electric field is zero. However, it experiences a torque given by

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{p}} \times \vec{\mathbf{E}}. \tag{26.19}$$

If we choose the electric field to be in the direction of $\hat{\mathbf{x}}$ and the dipole moment to be in the x-y plane making an angle θ with the electric field, we have

$$\vec{\boldsymbol{\tau}} = -\hat{\mathbf{z}}\, pE \sin\theta. \tag{26.20}$$

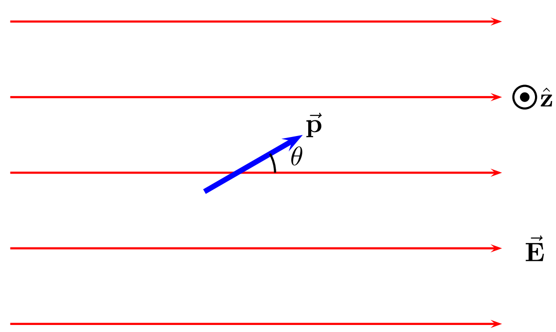

Figure 26.5: Electric dipole in a uniform electric field.

The change in potential energy of the electric dipole is the work done by the torque, while it rotates about the $\hat{\mathbf{z}}$ direction,

$$\Delta U = -\int_0^\theta \vec{\boldsymbol{\tau}} \cdot \hat{\mathbf{z}} d\theta = pE(1 - \cos\theta). \tag{26.21}$$

This energy can also be interpreted as the work done by the forces, while the dipole rotates about $\hat{\mathbf{z}}$,

$$\Delta U = -\int_0^\theta \vec{\mathbf{F}} \cdot d\vec{\mathbf{l}} = pE(1 - \cos\theta), \tag{26.22}$$

which uses the fact that the positive charge moves a distance $-a(1 - \cos\theta)/2$ and the negative charge moves a distance $a(1 - \cos\theta)/2$. The change in potential energy is only determined up to a constant. We fix this constant by choosing the potential energy to be zero when $\theta = \pi/2$. For this choice, we have the potential energy of an electric dipole in a uniform electric field given by

$$U = -\vec{\mathbf{p}} \cdot \vec{\mathbf{E}} = -pE\cos\theta. \tag{26.23}$$

As a check we verify that the torque on the electric dipole is indeed the negative derivative with respect to angle θ,

$$\vec{\boldsymbol{\tau}} = -\hat{\mathbf{z}}\frac{\partial}{\partial\theta}U = -\hat{\mathbf{z}}\,pE\sin\theta = \vec{\mathbf{p}} \times \vec{\mathbf{E}}. \tag{26.24}$$

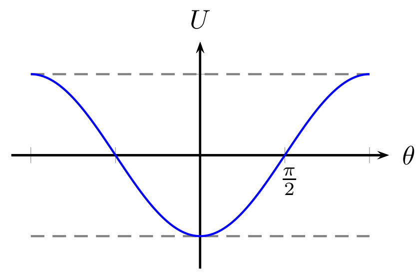

Figure 26.6: Potential energy of an electric dipole in a uniform electric field.

26.5 Dielectric material

Chapter 27

Current and resistance

We have learned that positive charges tend to move from a point of higher electrical potential to a point of lower electrical potential, and negative charges tend to do the opposite. This basic idea is at the heart of electrical circuits, which involves flow of electric charges. A traditional battery is a device that provides a (constant) potential difference, by moving charges against their natural tendency. The three basic electrical components will will discuss are: capacitor, resistor, and inductor.

27.1 Current

Flow of electric charges (in a conducting wire) is described by current,

$$I = \frac{dq}{dt}. \tag{27.1}$$

It is measured in units of Ampère=Coulomb/second. It is expressed in terms of the number density of charge carriers n, area of crosssection of the wire A, and drift velocity (speed of flow) v_d, as

$$I = neAv_d. \tag{27.2}$$

Lecture-Example 27.1: (Drift velocity)
Estimate the drift velocity in typical metals. Let us consider a current of $I = 1\,\text{A}$ passing through a copper wire with area of crosssection $A = \pi r^2 = \pi(1\,\text{mm})^2 \sim 3 \times 10^{-6}\,\text{m}^2$. Since Copper has one free electron per atom, density of $8.9\,\text{g/cm}^3$, and atomic weight of $63.5\,\text{g/mole}$, we estimate $n = 9 \times 10^{28}\,\text{atoms/m}^3$. (Avagadro's number is $6 \times 10^{23}\,\text{atoms/mole}$.) (Answer: $v_d = 2 \times 10^{-5}\,\text{m/s}$.)

- How much time does it take for an individual electron to begin from the light switch to the bulb that is connected by a $2\,\text{m}$ copper wire? (Answer: 28 hours.)
- To put on the light switch it is the flow that is relevant, very much like water arriving at the faucet instantly.

27.2 Resistance

Resistance in a wire is the opposition to the flow of charges. For standard materials it is proportional to the length of wire l, inversely proportional to area of crosssection A, in addition to it depending on the material specific property, the resistivity ρ. Together, we have

$$R = \frac{\rho l}{A}. \tag{27.3}$$

It is measured in units of Ohms=Volt/Ampère.

27.3 Ohm's law

The current I flowing through a resistor R is directly proportional to the potential difference across the resistor, for many materials. This is the statement of Ohm's law,

$$V = IR. \tag{27.4}$$

27.4 Power dissipated in a resistor

The power dissipated in a resistor is given by

$$P = IV = \frac{V^2}{R} = I^2 R. \tag{27.5}$$

Lecture-Example 27.2:
The average cost of electricity in the United States, for residential users, is about 0.15 USD/kWh (15 cents per kiloWatt-hour). At this rate your electricity bill for a month came out to be 50.00 USD. How much electric energy (in Joules) did you use in the month? (Answer: 1.2×10^9 J)

Chapter 28

Direct-current circuits

28.1 Resistors in series and parallel

A resistor when connected to a battery leads to a flow of current. The current I is decided by the resistance R and potential difference V across the resistor,

$$I = \frac{V}{R}. \tag{28.1}$$

Lecture-Example 28.1:
A resistor $R = 500\,\Omega$ is connected across a $10.0\,\text{V}$ battery. Determine the current in the circuit. (Answer: $20\,\text{mA}$.)

Resistors in series

Consider two resistors in series as described in Figure 28.1. Since the potential difference across the battery is distributed across the two resistors we deduce that

$$V = V_1 + V_2. \tag{28.2}$$

The current flowing both the resistors is the same,

$$I_1 = I_2, \tag{28.3}$$

because the channel for flow does not bifurcate. An equivalent resistor R_{eq} shown on the right side in Figure 28.1 is defined as a resistor that will pull the same amount of current from the battery. Thus, using $V_1 = I_1 R_1$, $V_2 = I_2 R_2$, and $V = I_{\text{eq}} R_{\text{eq}}$, in Eq. (28.2), we learn that

$$R_{\text{eq}} = R_1 + R_2. \tag{28.4}$$

We can further deduce that

$$\frac{V_1}{V_2} = \frac{R_1}{R_2}, \tag{28.5}$$

which turns out to be handy in the analysis of more complicated configurations.

Lecture-Example 28.2: (Resistors in series)
A potential difference $V = 10.0\,\text{V}$ is applied across a resistor arrangement with two resistances connected in series, $R_1 = 100.0\,\Omega$ and $R_2 = 200.0\,\Omega$.

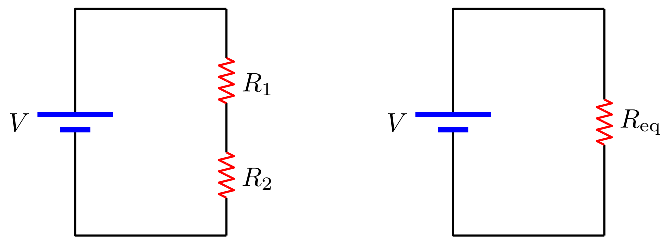

Figure 28.1: Resistors in series.

- Find the equivalent resistance. (Answer: $R_{eq} = 300.0\,\Omega$.)
- Find the currents I_1 and I_2 flowing through the resistors. (Answer: $I_1 = I_2 = 33.3\,\text{mA}$.)
- Find the voltages V_1 and V_2 across each of the resistors. (Answer: $V_1 = 3.33\,\text{V}$ and $V_2 = 6.67\,\text{V}$.)
- Find the power P_1 and P_2 dissipated in each of the resistors. (Answer: $P_1 = 111\,\text{mW}$ and $P_2 = 222\,\text{mW}$.)

Resistors in parallel

Consider two resistors in parallel as described in Figure 28.2. The potential difference across each resistor is identical,

$$V = V_1 = V_2. \tag{28.6}$$

The total current I that flows out of the battery distributes between the two resistors,

$$I = I_1 + I_2. \tag{28.7}$$

An equivalent resistor R_{eq} shown on the right side in Figure 28.2 is defined as a resistor that will pull the same amount of current from the battery. Thus, using $I_1 = V_1/R_1$, $I_2 = V_2/R_2$, and $I = V/R_{eq}$, in Eq. (28.7), we learn that

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}. \tag{28.8}$$

We can further deduce that

$$\frac{I_1}{I_2} = \frac{R_2}{R_1}, \tag{28.9}$$

which turns out to be handy in the analysis of more complicated configurations.

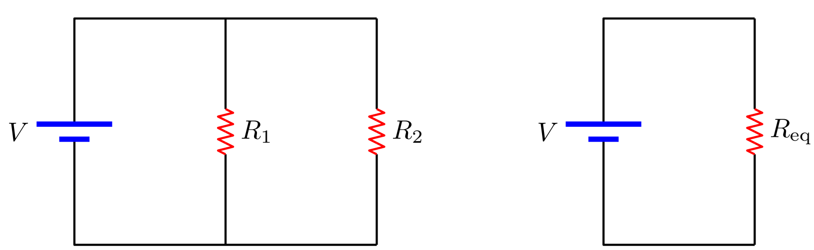

Figure 28.2: Resistors in parallel.

Lecture-Example 28.3: (Resistors in parallel)
A potential difference $V = 10.0\,\text{V}$ is applied across a resistor arrangement with two resistances connected in parallel, $R_1 = 100.0\,\Omega$ and $R_2 = 200.0\,\Omega$.

- Find the equivalent resistance. (Answer: $R_{\text{eq}} = 66.7\,\Omega$.)
- Find the voltages V_1 and V_2 across each of the resistors. (Answer: $V_1 = V_2 = 10.0\,\text{V}$.)
- Find the currents I_1 and I_2 flowing through each of the resistors. (Answer: $I_1 = 100\,\text{mA}$ and $I_2 = 50\,\text{mA}$.)
- Find the power P_1 and P_2 dissipated in each of the resistors. (Answer: $P_1 = 1.00\,\text{W}$ and $P_2 = 0.500\,\text{W}$.)

28.2 Kirchoff's circuit laws

Kirchoff's circuit laws are restatements of the the law of conservation of charge and the law of conservation of energy.

Kirchoff's junction rule

Kirchoff's junction rule states that at a junction in a circuit the sum of currents flowing into the junction is equal to the sum of currents flowing out of the junction. That is,

$$I_1 + I_2 + \ldots = 0. \tag{28.10}$$

Since currents are rate of change of charges, $I_i = dQ_i/dt$, we have

$$Q_1 + Q_2 + \ldots = \text{constant}, \tag{28.11}$$

which is the statement of conservation of charge.

Kirchoff's loop rule

Kirchoff's loop rule states that the sum of voltage drops around a closed loop in a circuit is zero. This is a consequence of Eq. (25.5). Since potential difference (voltage drops) is the change in energy per unit charge this is a restatement of conservation of energy.

Lecture-Example 28.4:
Determine the current in the circuit in Figure 28.3. Let $R_1 = 100\,\Omega$ and $R_2 = 200\,\Omega$.

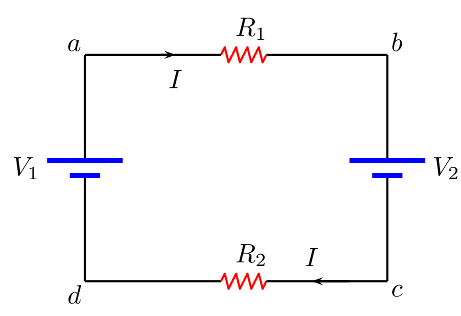

Figure 28.3: Lecture-Example 28.4

- Using Kirchoff's law for the loop $dabcd$ we have

$$+V_1 - IR_1 - V_2 - IR_2 = 0, \tag{28.12}$$

which leads to

$$I = \frac{V_1 - V_2}{R_1 + R_2}. \tag{28.13}$$

Evaluate the current for $V_1 = 10\,\text{V}$ and $V_2 = 20\,\text{V}$. (Answer: $I = -33.3\,\text{mA}$.)

- Evaluate the current for the case $V_1 \gg V_2$ with $V_1 = 10\,\text{V}$. (Answer: $I = 33.3\,\text{mA}$.) Evaluate the current for the case $V_1 \ll V_2$ with $V_2 = 10\,\text{V}$. (Answer: $I = -33.3\,\text{mA}$.) Evaluate the current for the case $V_1 = V_2$. (Answer: $I = 0$. Because there is no potential difference between point a and b.)

Lecture-Example 28.5:
Reanalyze the case of two resistors in parallel using Kirchoff's laws.

Lecture-Example 28.6:
Consider the circuit in Figure 28.4. Determine the currents in each of the resistors.

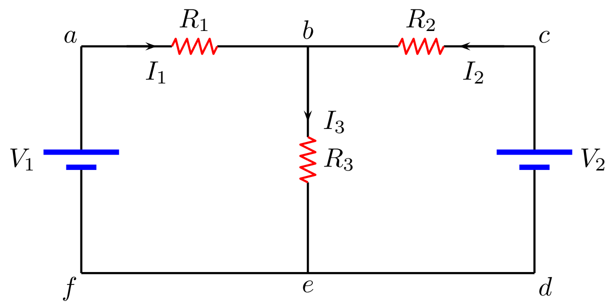

Figure 28.4: Lecture-Example 28.6

- Using Kirchoff's junction rule at the junction b we have

$$I_1 + I_2 = I_3. \tag{28.14}$$

Using Kirchoff's loop rule for the loop $fabef$ we have

$$+V_1 - I_1R_1 - I_3R_3 = 0. \tag{28.15}$$

Using Kirchoff's loop rule for the loop $bcdeb$ we have

$$+I_2R_2 - V_2 + I_3R_3 = 0. \tag{28.16}$$

This leads to

$$I_1 = \frac{(R_2 + R_3)V_1 - R_3V_2}{(R_1 + R_3)(R_2 + R_3) - R_3^2}, \tag{28.17a}$$

$$I_2 = \frac{(R_1 + R_3)V_2 - R_3V_1}{(R_1 + R_3)(R_2 + R_3) - R_3^2}. \tag{28.17b}$$

For $R_1 = 100\,\Omega$, $R_2 = 200\,\Omega$, $R_3 = 300\,\Omega$, $V_1 = 10\,\text{V}$, and $V_2 = 20\,\text{V}$, we find $I_1 = -9.09\,\text{mA}$, $I_2 = 45.5\,\text{mA}$, and $I_3 = 36.4\,\text{mA}$.

- Under what condition is the current I_1 in resistor R_1 zero? (Answer: $(R_2 + R_3)V_1 = R_3V_2$.) Under what condition is the current I_2 in resistor R_2 zero? (Answer: $(R_1 + R_3)V_2 = R_3V_1$.) Under what condition is the current I_3 in resistor R_3 zero? (Answer: $R_1V_2 + R_2V_1 = 0$, which is not possible for positive V_1 and V_2.)

Lecture-Example 28.7:
Consider the circuit in Figure 28.5. Determine the currents in each of the resistors. Let $R_1 = 100\,\Omega$, $R_2 = 200\,\Omega$, $V_1 = 10\,\text{V}$, and $V_2 = 20\,\text{V}$.

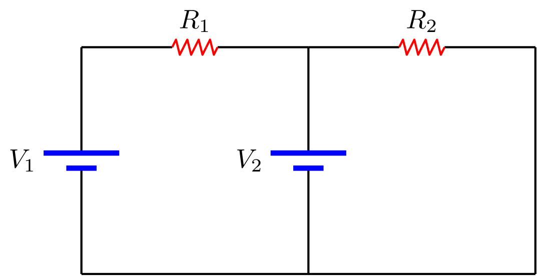

Figure 28.5: Lecture-Example 28.7

Lecture-Example 28.8: (Wheatstone bridge)
Consider the circuit in Figure 28.6. Show that the condition for no current flow through R_5 is

$$\frac{R_1}{R_2} = \frac{R_3}{R_4}. \tag{28.18}$$

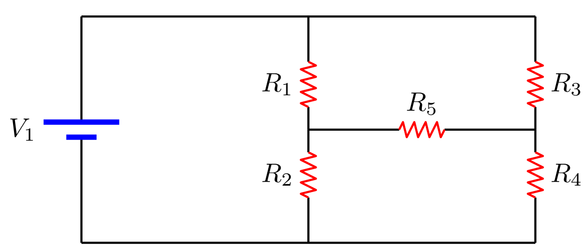

Figure 28.6: Lecture-Example 28.8

28.3 RC circuit

A resistor and capacitor in series constitutes a RC circuit. With a battery the circuit charges the capacitor and without the battery it discharges the capacitor.

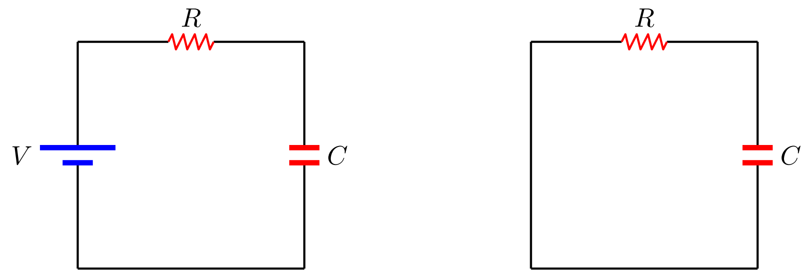

Figure 28.7: RC circuit: Charging and discharging of a capacitor.

Charging a capacitor

A resistor and a capacitor in series with a battery is governed by the equation, using Kirchoff's law,

$$V - IR - \frac{Q}{C} = 0. \tag{28.19}$$

Using $I = dQ/dt$ we can solve this differential equation with the initial condition $Q(0) = 0$ to yield

$$Q(t) = CV \left[1 - e^{-\frac{t}{RC}}\right]. \tag{28.20}$$

Thus, it takes infinite time to charge the capacitor to its maximum capacity, $Q(\infty) = CV$. Nevertheless, the rate at which the capacitor is charged is governed by $\tau = RC$, which is called the time constant of the circuit.

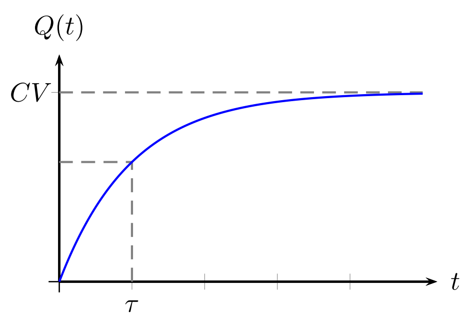

Figure 28.8: Charging of a capacitor

Lecture-Example 28.9: (Time constant)
Show that the amount of charge on the capacitor at time $t = \tau = RC$, during the process of charging of a capacitor is

$$Q(\tau) = CV\left(1 - \frac{1}{e}\right) \sim 0.632\, CV. \tag{28.21}$$

- Evaluate the time constant τ for the case $R = 1.0\,\text{M}\Omega$ and $C = 1.0\,\text{nF}$. (Answer: $\tau = 1.0\,\text{ms}$.)

Discharging a capacitor

A resistor and a capacitor in series without a battery is governed by the equation, using Kirchoff's law,

$$-IR - \frac{Q}{C} = 0. \tag{28.22}$$

Using $I = dQ/dt$ we can solve this differential equation with the initial condition $Q(0) = Q_0$ to yield

$$Q(t) = Q_0 e^{-\frac{t}{RC}}. \tag{28.23}$$

Thus, it takes infinite time to discharge the capacitor completely. The rate at which the capacitor is discharged is again governed by the time constant $\tau = RC$.

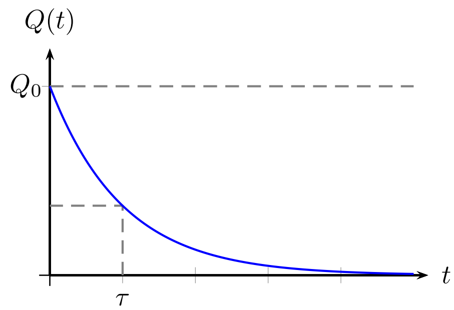

Figure 28.9: Discharging of a capacitor

Lecture-Example 28.10: (Time constant)
Show that the amount of charge on the capacitor at time $t = \tau = RC$, during the process of discharging of a capacitor is

$$Q(\tau) = Q_0 \frac{1}{e} \sim 0.368\, Q_0. \tag{28.24}$$

Chapter 29

Magnetic force

29.1 Magnetic field

The concepts introduced in electrostatics can be summarized in the following symbolic form:

$$\text{Charge } q_1 \quad \rightarrow \quad \text{Electric field } (\vec{\mathbf{E}}_1) \quad \rightarrow \quad \text{Charge } q_2 \text{ feels a force } \vec{\mathbf{F}}_{21} = q_2\vec{\mathbf{E}}_1$$

That is, a charge q_1 creates an electric field $\vec{\mathbf{E}}_1$ which exerts a force $\vec{\mathbf{F}}_{21}$ on another charge q_2. A moving charge, in addition to the above, leads to a new phenomenon. A moving charge creates a magnetic field which exerts a force on another moving charge. This is summarized in the form:

$$\text{Moving charge } q_1\vec{\mathbf{v}}_1 \quad \rightarrow \quad \text{Magnetic field } (\vec{\mathbf{B}}_1) \quad \rightarrow \quad \text{Moving charge } q_2\vec{\mathbf{v}}_2 \text{ feels a force } \vec{\mathbf{F}}_{21} = q_2\vec{\mathbf{v}}_2 \times \vec{\mathbf{B}}_1$$

Thus, a charge q moving with velocity $\vec{\mathbf{v}}$, represented by

$$q\vec{\mathbf{v}} \tag{29.1}$$

or the corresponding current due to the movement of the charge, is a source of magnetic field. A manifestation of this phenomena at the microscopic level is seen in the interaction of two magnets, where the magnetic field due to one magnet exerts a force on the second magnet.

The Magnetic field is measured in units of Tesla=N·s/C·m. The common magnetic fields we come across is listed in Table 29.1.

29.2 Vector product

Vector product (or the cross product) of two vectors

$$\vec{\mathbf{A}} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}, \tag{29.2a}$$

$$\vec{\mathbf{B}} = B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}} + B_z\,\hat{\mathbf{k}}, \tag{29.2b}$$

10^8 T	magnetic field of a neutron star
10^2 T	strength of a laboratory magnet
10^1 T	medical MRI
10^0 T	a neodymium magnet
10^{-3} T	a refrigerator magnet
10^{-4} T	strength on surface of Earth
10^{-12} T	human brain

Table 29.1: Orders of magnitude (magnetic field)

is given by

$$\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_y B_z - A_z B_y)\,\hat{\mathbf{i}} - (A_x B_z - A_z B_x)\,\hat{\mathbf{j}} + (A_x B_y - A_y B_x)\,\hat{\mathbf{k}} \tag{29.3a}$$

$$= AB \sin\theta\,\hat{\mathbf{n}} \tag{29.3b}$$

where θ is the angle between the two vectors. The vector product measures the area associated with the two vectors. The direction of the vector product $\vec{\mathbf{C}}$ is given by the right-hand rule. The right-hand rule is a mnemonic that associates the thumb to the vector $\vec{\mathbf{A}}$, the fingers to the vector $\vec{\mathbf{B}}$, such that the vector $\vec{\mathbf{C}}$ is in the direction facing the palm of the right hand.

In discussions concerning three dimensions we often have quantities pointing in and out of a plane. We shall use the notation $\odot$ to represent a direction coming out of the plane, and $\otimes$ to represent a direction going into the plane. As a mnemonic one associates the dot with the tip of an arrow coming out of the page and the cross with the feathers of an arrow going into the page.

29.3 Magnetic force

The force on a charge q moving with velocity $\vec{\mathbf{v}}$ in a magnetic field $\vec{\mathbf{B}}$ is symbolically given by

$$\vec{\mathbf{F}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}. \tag{29.4}$$

The magnitude and direction of the magnetic force $\vec{\mathbf{F}}$ is given by the right-hand rule for vector product. The right-hand rule applies to a positive charge. For a negative charge the direction of force is flipped.

Lecture-Example 29.1:
A proton and an electron enters a region containing a magnetic field going into the page, $\vec{\mathbf{B}} = -2.0\,\hat{\mathbf{z}}\,\mathrm{T}$. Let the velocity of both the particles while they enter the region be to the right, $\vec{\mathbf{v}} = 3.0 \times 10^5\,\hat{\mathbf{x}}\,\mathrm{m/s}$.

- Determine the magnitude of the magnetic force on the proton and the electron.
- Determine the direction of the magnetic force on the proton and the electron, using the right-hand rule.
- Determine the corresponding accelerations experienced the proton and the electron.

29.4 Motion of a charged particle in a uniform magnetic field

Using Newton's law, $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, we have the equation of motion for a charged particle in a uniform magnetic field to be

$$\frac{d\vec{\mathbf{v}}}{dt} = \frac{q}{m}\vec{\mathbf{v}} \times \vec{\mathbf{B}}. \tag{29.5}$$

In a uniform magnetic $\vec{\mathbf{B}}$, if the velocity of a particle $\vec{\mathbf{v}}$ is perpendicular to the direction of the magnetic field, the direction of the acceleration of the particle is always perpendicular to the velocity of the particle and to the magnetic field. Further, for the case of uniform magnetic field the magnitude of the acceleration remains constant. These are the requirements for a particle to move in a circle with uniform speed. Thus, using Newton's law, $F = ma$, for circular motion, we have

$$qvB = m\frac{v^2}{R}, \tag{29.6}$$

where R is the radius of the circle and ω is the angular frequency of the rotational motion, such that where $v = \omega R$. We learn that the particle particle goes around the magnetic field at an angular frequency, the cyclotron frequency, given by

$$\omega = \frac{q}{m}B, \tag{29.7}$$

which depends on the charge to mass ratio of the particle.

For the more general case of the velocity not being perpendicular to the magnetic field the particle drifts in the direction of the magnetic field while moving in circles, the path covered being helical.

Lecture-Example 29.2: (Circular motion)
Motion of a charged particle of mass m and charge q in a uniform magnetic field $\mathbf{B}$ is governed by

$$m\frac{d\mathbf{v}}{dt} = q\,\mathbf{v}\times\mathbf{B}. \tag{29.8}$$

Choose $\mathbf{B}$ along the z-axis and solve this vector differential equation to determine the position $\mathbf{x}(t)$ and velocity $\mathbf{v}(t)$ of the particle as a function of time, for initial conditions

$$\mathbf{x}(0) = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{29.9a}$$

$$\mathbf{v}(0) = 0\,\hat{\mathbf{i}} + v_0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}. \tag{29.9b}$$

Verify that the solution describes a circle with center at position $R\,\hat{\mathbf{i}}$.

Lecture-Example 29.3: (Northern lights)
A proton and an electron are moving in circles around a magnetic field of $B = 1.0\times 10^{-6}\,\mathrm{T}$.

- Determine the cyclotron frequency for the proton and the electron.
  (Answer: $\omega_p = 96\,\mathrm{rad/s}$, $\omega_e = 1.8\times 10^5\,\mathrm{rad/s}$.)
- If the particles are moving with uniform speed $v = 2.0\times 10^6\,\mathrm{m/s}$, determine the radius of the circles describing their path. (Answer: $R_p = 21\,\mathrm{km}$, $R_e = 11\,\mathrm{m}$.)
- Aurora Borealis (northern lights) and Aurora Australis (southern lights) is a spectacular display of light shimmering across the night sky, often observed around magnetic poles of the Earth, when charged particles emitted by the Sun and guided along by the magnetic field of the Earth enter the atmosphere. Check out an animation of this phenomenon as seen from space, released by NASA Earth Observatory,

  Aurora Australis on 2005 Sep 11,

  which to an observer on Earth would appear as a curtain of shimmering light.

Lecture-Example 29.4: (Bubble chamber)
Refer to the following tutorial at CERN: How to read Bubble Chamber pictures.

Lecture-Example 29.5: (Velocity selector)
The electric field and the magnetic field both deflect charged particles due to the respective forces. In a velocity selector these forces are exactly balanced for particles moving with a particular velocity which go through undeviated. Show that the velocity of a velocity selector is determined by

$$v = \frac{E}{B}. \tag{29.10}$$

- Determine the velocity selected by a velocity selector consisting of an electric field of $E = 3.0 \times 10^5$ N/C and a magnetic field of $B = 1.5$ T. (Answer: $v = 2.0 \times 10^5$ m/s.)

Lecture-Example 29.6: (Applications)

- Mass spectrometer
- Hall effect
- Cyclotron
- Cathode ray tube

29.5 Magnetic force on a current carrying wire

Using the fact that a current carrying wire involves the motion of positive positive charges we realize that the wire will experience a magnetic force in a magnetic field. Identifying the relation

$$dq\,\vec{\mathbf{v}} = dq\,\frac{d\vec{\mathbf{l}}}{dt} = I d\vec{\mathbf{l}}, \tag{29.11}$$

where I is the current in the wire, we derive the force on a current carrying wire to be given by the line integral

$$\vec{\mathbf{F}} = \int_C I d\vec{\mathbf{l}} \times \vec{\mathbf{B}}, \tag{29.12}$$

C being the curve that specifies the shape of the wire. The direction of the force is given using the right-hand rule with the thumb in the direction of current.

Lecture-Example 29.7:
A loop in the shape of a right triangle, carrying a current I, is placed in a magnetic field. (Choose $\hat{\mathbf{z}}$ to be out of the page.)

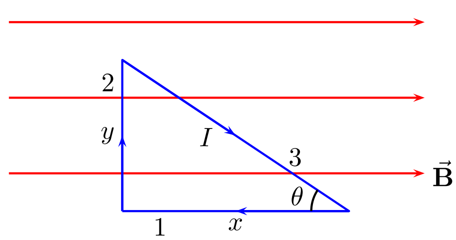

Figure 29.1: Lecture-Example 29.7

- The force on side 1 is given by
$$\vec{\mathbf{F}}_1 = IBx \sin 180^\circ\,\hat{\mathbf{z}} = 0\,\hat{\mathbf{z}}. \tag{29.13}$$
The force on side 2 is given by
$$\vec{\mathbf{F}}_2 = -IBy \sin 90^\circ\,\hat{\mathbf{z}} = -IBy\,\hat{\mathbf{z}}. \tag{29.14}$$
The force on side 3 is given by, using $\sin\theta = y/\sqrt{x^2 + y^2}$,
$$\vec{\mathbf{F}}_3 = IB\sqrt{x^2 + y^2} \sin\theta\,\hat{\mathbf{z}} = IBy\,\hat{\mathbf{z}}. \tag{29.15}$$

- Show that the total force on the triangle is zero.

Lecture-Example 29.8:
A loop in the shape of a right triangle, carrying a current I, is placed in a magnetic field. (Choose $\hat{\mathbf{z}}$ to be out of the page.)

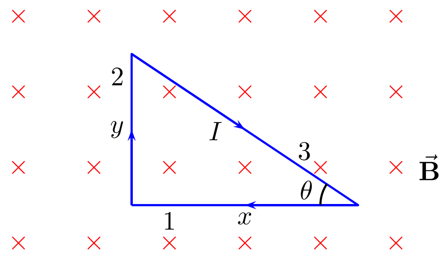

Figure 29.2: Lecture-Example 29.8

- The force on side 1 is given by
$$\vec{\mathbf{F}}_1 = -IBx\sin 90^\circ\,\hat{\mathbf{y}} = -IBx\,\hat{\mathbf{y}}. \tag{29.16}$$
The force on side 2 is given by
$$\vec{\mathbf{F}}_2 = -IBy\sin 90^\circ\,\hat{\mathbf{x}} = -IBy\,\hat{\mathbf{x}}. \tag{29.17}$$
The force on side 3 is given by, using $\sin\theta = y/\sqrt{x^2+y^2}$,
$$\vec{\mathbf{F}}_3 = IB\sqrt{x^2+y^2}\sin\theta\,\hat{\mathbf{x}} + IB\sqrt{x^2+y^2}\cos\theta\,\hat{\mathbf{y}} = IBy\,\hat{\mathbf{x}} + IBx\,\hat{\mathbf{y}}. \tag{29.18}$$
- Show that the total force on the triangle is zero.

Lecture-Example 29.9:
A loop in the shape of a semi circle of radius R, carrying a current I, is placed in a magnetic field. (Choose $\hat{\mathbf{z}}$ to be out of the page.)

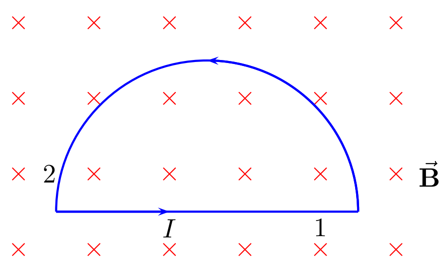

Figure 29.3: Lecture-Example 29.9

- Show that the force on side 1 is given by
$$\vec{\mathbf{F}}_1 = 2IRB\,\hat{\mathbf{y}}. \tag{29.19}$$
Show that the force on side 2 is given by
$$\vec{\mathbf{F}}_2 = -2IRB\,\hat{\mathbf{y}}. \tag{29.20}$$
- Show that the total force on the loop is zero.

29.6 Magnetic moment of a current carrying loop

The magnetic moment $\vec{\boldsymbol{\mu}}$ associated with a (planar) current carrying loop of wire is

$$\vec{\boldsymbol{\mu}} = NIA\,\hat{\mathbf{n}}, \tag{29.21}$$

where I is the current in the wire, N is the number of turns in the loop, and A is the area of the loop. The direction of the magnetic moment, represented by $\hat{\mathbf{n}}$, is perpendicular to the plane constituting the loop and is given by the right-hand rule. An arbitrary shaped loop that is not planar can be constructed out of infinitely small planar loops.

A magnet is interpreted to have a North and South pole, in the Gilbert model. In the Ampère model the magnetic field due to a magnet is due to microscopic current loops. The magnetic moment of a magnet characterizes the strength of a magnetic field produced by the magnet.

Force

The total force on a current carrying loop in a uniform magnetic field is zero, that is,

$$\oint I d\vec{\mathbf{l}} \times \vec{\mathbf{B}} = 0. \tag{29.22}$$

Torque

The magnitude of the torque on a current carrying loop, or just a magnetic moment $\vec{\boldsymbol{\mu}}$, in a uniform magnetic field $\vec{\mathbf{B}}$, is

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\mathbf{B}}. \tag{29.23}$$

The direction is such that the magnetic moment tries to align with the magnetic field.

Lecture-Example 29.10:
A loop in the shape of a rectangle, carrying a current I, is placed in a magnetic field. Let the plane of the loop be perpendicular to the magnetic field $\vec{\mathbf{B}} = -B\,\hat{\mathbf{z}}$.

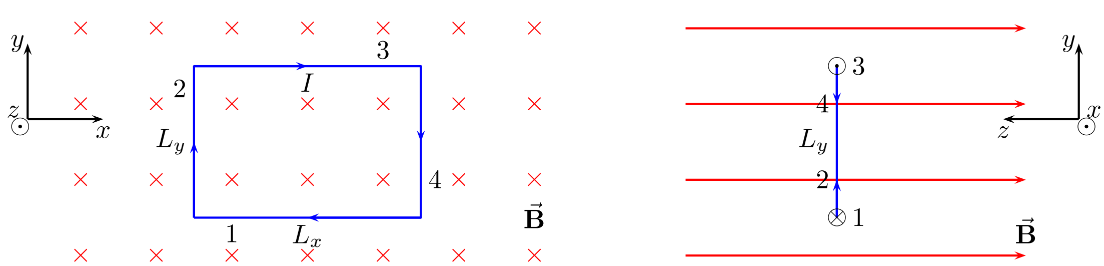

Figure 29.4: Lecture-Example 29.10

- The force on side 1 is given by

$$\vec{\mathbf{F}}_1 = -\hat{\mathbf{y}}\,IL_xB\sin 90^\circ = -\hat{\mathbf{y}}\,IL_xB. \tag{29.24}$$

The force on side 2 is given by

$$\vec{\mathbf{F}}_2 = -\hat{\mathbf{x}}\,IL_yB\sin 90^\circ = -\hat{\mathbf{x}}\,IL_yB. \tag{29.25}$$

The force on side 3 is given by

$$\vec{\mathbf{F}}_3 = +\hat{\mathbf{y}}\, I L_x B \sin 90^\circ = +\hat{\mathbf{y}}\, I L_x B. \tag{29.26}$$

The force on side 4 is given by

$$\vec{\mathbf{F}}_4 = +\hat{\mathbf{x}}\, I L_y B \sin 90^\circ = +\hat{\mathbf{x}}\, I L_y B. \tag{29.27}$$

- Show that the total force on the rectangle is zero.
- Show that the total torque on the rectangle is zero.

Lecture-Example 29.11:
A loop in the shape of a rectangle, carrying a current I, is placed in a magnetic field. Let the normal to the plane of the loop make an angle θ with respect to the magnetic field $\vec{\mathbf{B}} = -B\,\hat{\mathbf{z}}$.

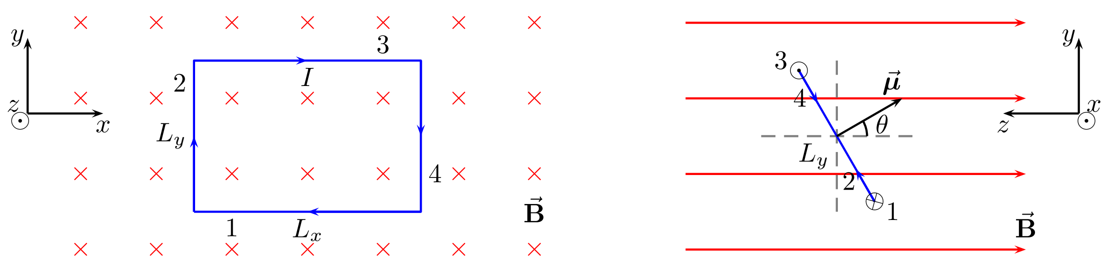

Figure 29.5: Lecture-Example 29.11

- The force on side 1 is given by

$$\vec{\mathbf{F}}_1 = -\hat{\mathbf{y}}\, I L_x B \sin 90^\circ = -\hat{\mathbf{y}}\, I L_x B. \tag{29.28}$$

The force on side 2 is given by

$$\vec{\mathbf{F}}_2 = -\hat{\mathbf{x}}\, I L_y B \sin(90^\circ + \theta) = -\hat{\mathbf{x}}\, I L_y B \cos\theta. \tag{29.29}$$

The force on side 3 is given by

$$\vec{\mathbf{F}}_3 = +\hat{\mathbf{y}}\, I L_x B \sin 90^\circ = +\hat{\mathbf{y}}\, I L_x B. \tag{29.30}$$

The force on side 4 is given by

$$\vec{\mathbf{F}}_4 = +\hat{\mathbf{x}}\, I L_y B \sin(90^\circ - \theta) = +\hat{\mathbf{x}}\, I L_y B \cos\theta. \tag{29.31}$$

- Show that the total force on the rectangle is zero.
- Show that the total torque on the rectangle is

$$\tau = \frac{L_y}{2} F_3 \sin\theta + \frac{L_y}{2} F_1 \sin\theta = \mu B \sin\theta. \tag{29.32}$$

Chapter 30

Magnetic field

30.1 Biot-Savart law

The magnetic field $d\vec{\mathbf{B}}$ generated by a current I through a line element $d\vec{\mathbf{l}}$,

$$I d\vec{\mathbf{l}}, \tag{30.1}$$

is given by the Biot-Savart law,

$$d\vec{\mathbf{B}} = \frac{\mu_0}{4\pi} \frac{I d\vec{\mathbf{l}} \times \hat{\mathbf{r}}}{r^2}. \tag{30.2}$$

The constant μ_0 is the permeability of vacuum,

$$\mu_0 = 4\pi \times 10^{-7} \, \frac{\text{T}\cdot\text{m}}{\text{A}}, \tag{30.3}$$

that describes the magnetic property of vacuum.

Lecture-Example 30.1: (Speed of light)
Evaluate

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}. \tag{30.4}$$

Answer: $c = 2.998 \times 10^8$ m/s, up to four significant digits. This is the speed of light!

Lecture-Example 30.2: (A straight segment of wire)
How that the magnetic field due to a straight segment of wire at a distance r from the wire is given by

$$\vec{\mathbf{B}} = \hat{\boldsymbol{\phi}} \, \frac{\mu_0 I}{4\pi r} (\sin\theta_1 + \sin\theta_2), \tag{30.5}$$

where the angles θ_1 and θ_2 specifies the observation point with respect to the ends of the wire. See Figure 30.1. The direction of the magnetic field $\hat{\boldsymbol{\phi}}$ is given by the right-hand rule and is tangential to circles around the wire.

- As a special case, we have the magnetic field due to an infinitely long wire, $\theta_1 = \theta_2 = \pi/2$, as

$$\vec{\mathbf{B}} = \hat{\boldsymbol{\phi}} \, \frac{\mu_0 I}{2\pi r}. \tag{30.6}$$

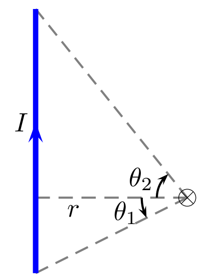

Figure 30.1: A straight segment of wire.

Lecture-Example 30.3: (A circular segment of wire)
Show that the magnetic field due a circular segment of wire, at the center of circle, is given by

$$\vec{\mathbf{B}} = \hat{\mathbf{z}}\,\frac{\mu_0 I}{4\pi R}\theta, \tag{30.7}$$

where angle θ is the angular measure of the segment. See Figure 30.2. The direction of the magnetic field $\hat{\mathbf{n}}$ is given by the right-hand rule and is perpendicular to the plane containing the segment of wire. As a special

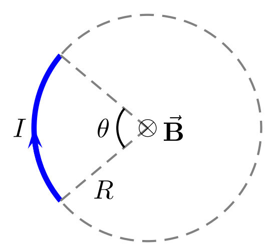

Figure 30.2: A straight segment of wire.

case, we have the magnetic field due to a circular loop of wire, $\theta = 2\pi$, at the center of the loop, as

$$\vec{\mathbf{B}} = \hat{\mathbf{z}}\,\frac{\mu_0 I}{2R}. \tag{30.8}$$

Lecture-Example 30.4: (Magnetic field on the symmetry axis of a circular wire)
Show that the magnetic field on the symmetry axis of a circular loop of wire carrying current I is given by

$$\vec{\mathbf{B}} = \hat{\mathbf{z}}\,\frac{\mu_0 I}{2}\frac{a^2}{(a^2+h^2)^{\frac{3}{2}}}, \tag{30.9}$$

where a is the radius of the circular loop. The direction of the magnetic field $\hat{\mathbf{n}}$ is given by the right-hand rule and is perpendicular to the plane containing the segment of wire.

- For the case $h \ll a$ this leads to the expression for the magnetic field of a circular at the center of the loop.

- For the case $a \ll h$ we have
$$\vec{\mathbf{B}} = \hat{\mathbf{z}}\,\frac{\mu_0}{2\pi}\frac{\mu}{h^3}, \qquad \mu = IA, \qquad A = \pi a^2, \tag{30.10}$$
where μ is the magnetic moment of the loop, and A is the area of the loop.

Lecture-Example 30.5: (Solenoid)
A solenoid is a coil in the form of a helix. For a closely wound coil we can model a solenoid as closely packed circular coils. Using the expression for the magnetic field due to a circular loop, show that the magnetic field along the axis of a solenoid is given by
$$\vec{\mathbf{B}} = \hat{\mathbf{z}}\,\mu_0 I n, \tag{30.11}$$
where $n = N/L$ is the number of turns per unit length. The direction of the magnetic field $\hat{\mathbf{z}}$ is given by the right-hand rule and is perpendicular to the plane containing the segment of wire.

- In general the magnetic field due to a solenoid is given by, (which will be proved using Ampère's law later,)
$$\vec{\mathbf{B}} = \begin{cases} \hat{\mathbf{z}}\,\mu_0 I n, & \text{inside}, \\ 0, & \text{outside}. \end{cases} \tag{30.12}$$
Observe that a solenoid creates a uniform magnetic field inside the solenoid.

Lecture-Example 30.6: A steady current I flows through a wire shown in Fig. 30.3. Show that the magnitude and direction of magnetic field at point P is
$$B = \frac{\mu_0 I}{4\pi a}\left(\frac{2}{2} + \frac{2}{2} + \frac{2\pi}{2}\right) \tag{30.13}$$
coming out of the page.

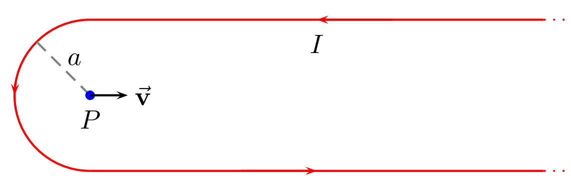

Figure 30.3: Lecture-Example 30.6

- Determine the magnitude and direction of the magnetic field for $I = 1.0\,\text{A}$ and $a = 10.0\,\text{cm}$.
- Determine the magnitude and direction of the magnetic force on a proton moving with velocity $v = 2.0 \times 10^6\,\text{m/s}$, to the right, while it is passing the point P.

Lecture-Example 30.7:
A steady current I flows through a wire shown in Fig. 30.4. Show that the magnitude and direction of magnetic field at point P is
$$B = \frac{\mu_0 I}{4\pi a}\left(\frac{2}{2} + \frac{2}{2} + \frac{2\pi}{4}\right) \tag{30.14}$$
going into the page.

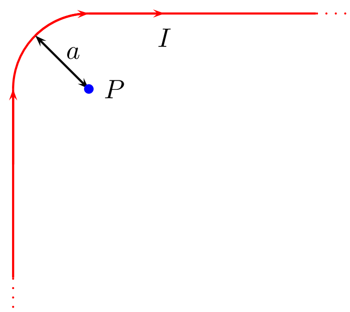

Figure 30.4: Lecture-Example 30.7

- Determine the magnitude and direction of the magnetic field for $I = 1.0\,\mathrm{A}$ and $a = 10.0\,\mathrm{cm}$.
- Determine the magnitude and direction of the magnetic force on a proton moving with velocity $v = 2.0 \times 10^6\,\mathrm{m/s}$, to the right, while it is passing the point P.

Lecture-Example 30.8: A steady current I flows through a wire in the shape of a square of side L, shown in Fig. 30.5. Show that the magnitude and direction of the magnetic field at the center of the square, P, is

$$B = \frac{\mu_0 I}{\pi L} \frac{4}{\sqrt{2}} \tag{30.15}$$

going out of the page.

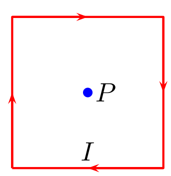

Figure 30.5: Lecture-Example 30.8

- Repeat this for an equilateral triangle, and a polygon.

Lecture-Example 30.9:
Figure 30.6 shows two current carrying wires, separated by a distance D. The directions of currents, either going into the page or coming out of the page, are shown in the figure. Determine the point $\times$ where the magnetic field is exactly zero.

- Answer:

$$x = \frac{D}{\left(1 + \frac{I_2}{I_1}\right)}. \tag{30.16}$$

Determine x if $I_1 = 2.0\,\mathrm{A}$, $I_2 = 6.0\,\mathrm{A}$, and $D = 10.0\,\mathrm{cm}$. (Answer: 2.5 A.)

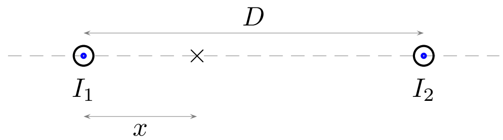

Figure 30.6: Lecture-Example 30.9

- How does your answer change if the direction of currents in either or both the wires were changed?

Lecture-Example 30.10:
Figure 30.7 shows two current carrying wires, in a plane. The directions of currents, either going into the page or coming out of the page, are shown in the figure. Determine the magnitude and direction of the magnetic field at the point $\times$, the origin. Let $I_1 = 1.0\,\text{A}$, $I_2 = 2.0\,\text{A}$, $x = 12\,\text{cm}$, and $y = 8.0\,\text{cm}$.

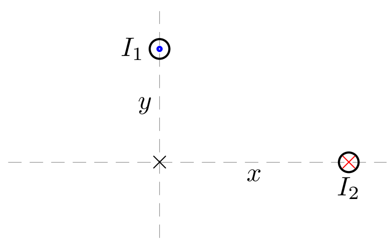

Figure 30.7: Lecture-Example 30.10

- The magnetic field at the origin due to the individual wires is

$$\vec{\mathbf{B}}_1 = \frac{\mu_0 I_1}{2\pi y}\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}}, \tag{30.17a}$$

$$\vec{\mathbf{B}}_2 = 0\,\hat{\mathbf{i}} + \frac{\mu_0 I_2}{2\pi x}\,\hat{\mathbf{j}}. \tag{30.17b}$$

The total magnetic field is given as

$$\vec{\mathbf{B}}_{\text{tot}} = \vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2. \tag{30.18}$$

Answer: $\vec{\mathbf{B}}_1 = \hat{\mathbf{i}}\,2.5\,\mu\text{T}$ and $\vec{\mathbf{B}}_2 = \hat{\mathbf{j}}\,3.3\,\mu\text{T}$. Magnitude $|\vec{\mathbf{B}}_{\text{tot}}| = 4.1\,\mu\text{T}$ makes an angle of 53° counter-clockwise with respect to x-axis.

- How does your answer change if the direction of currents in either or both the wires were changed?

30.2 Magnetic force between two parallel current carrying wires

If we have two parallel current carrying wires, each of the wires generates a magnetic field around it, which in turn exerts a force on the other wire. For currents I_1 and I_2 in the wires separated by a distance r we have the force per unit length on the wires given by

$$\frac{F}{L} = \frac{\mu_0 I_1 I_2}{2\pi r}. \tag{30.19}$$

The direction of the force is such that the wires attract if the current are in the same direction, and vice versa. We say, like currents attract and unlike current repel.

Lecture-Example 30.11:
Two infinitely long parallel wires, carrying currents $I_1 = 1.0\,\text{A}$ and $I_2 = 2.0\,\text{A}$ in the same direction, are separated by a distance $r = 10\,\text{cm}$.

- Determine the magnitude and direction of the magnetic field $\vec{\mathbf{B}}_1$ generated by the current I_1 at the position of current I_2. (Answer: $B_1 = 2.0\,\mu\text{T}$.) Determine the magnitude and direction of the force exerted by the magnetic field $\vec{\mathbf{B}}_1$ on the wire with current I_2. (Answer: $4.0\,\mu\text{N}$.)
- How will the answer differ if the currents are in opposite directions?

Lecture-Example 30.12:
A rectangular loop of wire carrying current $I_2 = 2.0\,\text{A}$ is placed near an infinitely long wire carrying current $I_1 = 1.0\,\text{A}$, such that two of the sides of the rectangle are parallel to the current I_1. Let the distances be $a = 5.0\,\text{cm}$, $b = 4.0\,\text{cm}$, and $l = 10.0\,\text{cm}$.

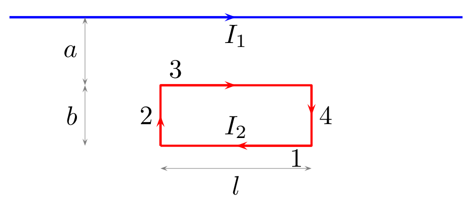

Figure 30.8: Lecture-Example 30.12

- Determine the force on side '1' of the loop. (Answer: $0.44\,\mu\text{N}$ away from the current carrying wire I_1.) Determine the force on side '3' of the loop. (Answer: $0.80\,\mu\text{N}$ towards the current carrying wire I_1.) Further, show that the combined force on side '2' and '3' is zero. Determine the magnitude and direction of the total force on the loop. (Answer: $0.36\,\mu\text{N}$ towards the current carrying wire I_1.)
- How does your analysis change if either of the currents were reversed?

30.3 Ampère's law

Ampère's law states that the line integral of the magnetic field $\vec{\mathbf{B}}$ along a closed path is completely determined by the total current I_{en} passing through the closed path,

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 I_{\text{en}}. \tag{30.20}$$

Lecture-Example 30.13: (Magnetic field due to an infinitely long current carrying wire)

Using the symmetry of an infinitely long straight wire, presuming the magnetic field to be circular, derive the magnetic field around the wire using Ampère's law,

$$\vec{\mathbf{B}} = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{2\pi r}. \tag{30.21}$$

Lecture-Example 30.14: (Solenoid)
Using Ampère's law show that the magnetic field due to a solenoid is given by,

$$\vec{\mathbf{B}} = \begin{cases} \hat{\mathbf{z}}\,\mu_0 I n, & \text{inside,} \\ 0, & \text{outside.} \end{cases} \tag{30.22}$$

Chapter 31

Faraday induction

31.1 Magnetic flux

Flux associated with the magnetic field $\vec{\mathbf{B}}$ across an infinitesimal area $d\mathbf{A}$ is defined as

$$d\Phi_B = \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}. \tag{31.1}$$

Flux associated with the magnetic field $\vec{\mathbf{B}}$ across a surface area S is then given by

$$\Phi_B = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}. \tag{31.2}$$

Gauss's law for magnetism states that the magnetic flux across a closed surface is zero,

$$\oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0, \tag{31.3}$$

which implies the absence of an isolated magnetic monopole, or the magnetic charge. In other words it states that the north pole and the south pole of a bar magnet can not be separated.

Lecture-Example 31.1: A square loop of wire consisting of a single turn is perpendicular to a uniform magnetic field. The square loop is then re-formed into a circular loop and is also perpendicular to the same magnetic field. Determine the ratio of the flux through the square loop to the flux through the circular loop. (Answer: $\pi/4$.)

31.2 Faraday's law of induction

Faraday's law of induction states that the negative rate of change of magnetic flux passing a loop of wire induces an effective voltage in the loop, which in turn generates a current in the loop,

$$IR = \Delta V_{\text{eff}} = -N \frac{d\phi_B}{dt}, \tag{31.4}$$

where N is the number of loops.

Lecture-Example 31.2:
Consider a straight wire of length $L = 1.0\,\text{m}$ moving with velocity $v = 30.0\,\text{m/s}$ in the region of a uniform

magnetic field $B = 2.0 \times 10^{-5}$ T. Determine the potential difference induced between the ends of the wire. (Answer: 0.60 mV.)

Lecture-Example 31.3: (Induction due to change in area)
Figure 31.1 shows a conducting a rod being pulled along horizontal, frictionless, conducting rails at a constant speed v. A uniform magnetic field $\mathbf{B}$ fills the region in which the rod moves. Let $l = 10$ cm, $v = 5.0$ m/s, $B = 1.2$ T, and $R = 0.40\,\Omega$.

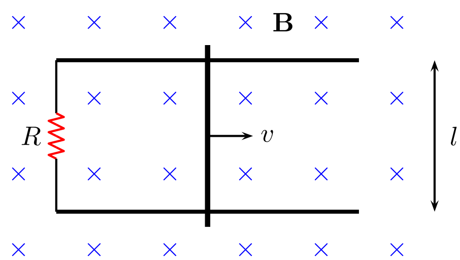

Figure 31.1: Lecture-Example 31.3

- Is the magnetic flux in the loop increasing or decreasing? What is the direction of the induced current in the loop?
- Show that the magnitude of the induced current in the loop is given by

$$I = \frac{Blv}{R}. \tag{31.5}$$

  Show that this induced current feels a magnetic force of

$$F_B = \frac{B^2 l^2 v}{R}. \tag{31.6}$$

  Determine the power delivered to the resistance due to the induced current is

$$P = \frac{B^2 l^2 v^2}{R}. \tag{31.7}$$

- How does the analysis change if the direction of velocity is reversed?

Lecture-Example 31.4:
Figure 31.2 shows five snapshots of a rectangular coil being pushed across the dotted region where there is a uniform magnetic field directed into the page. Outside of this region the magnetic field is zero.

- Determine the direction of induced current in the loop at each of the five instances in the figure.
- Determine the direction force on the loop due to the induced current in each of the five instances in the figure.

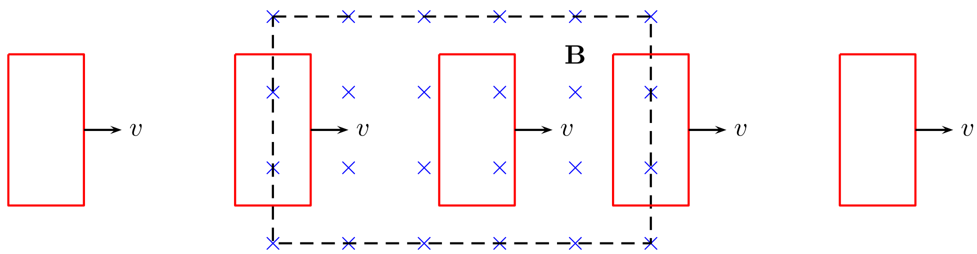

Figure 31.2: Lecture-Example 31.4

Lecture-Example 31.5: (Induction due to change in magnetic field)
A loop of wire having a resistance $R = 100.0\,\Omega$ is placed in a magnetic field whose magnitude is changing in time as

$$B = B_0 e^{-\frac{t}{\tau}}, \tag{31.8}$$

described in Figure 31.3, where τ is the interpreted as the time constant of the decay in the magnetic field. The direction of the magnetic field is normal to the plane of the loop. The loop of wire consists of 100 turns and has an area of $A = 25 \times 10^{-4}\,\text{m}^2$. Let $B_0 = 0.20\,\text{T}$ and $\tau = 0.10\,\text{ms}$.

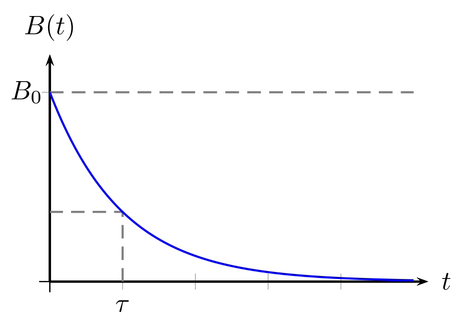

Figure 31.3: Lecture-Example 31.5

- Show that the induced current in the loop is given by

$$I = \frac{BA}{\tau R}. \tag{31.9}$$

Lecture-Example 31.6: (A simple transformer)
Consider two coils wound on the same cylinder such that the flux through both the coils is the same, such that

$$\frac{d\Phi_1}{dt} = \frac{d\Phi_2}{dt}. \tag{31.10}$$

Thus, derive the ratio of the voltages in the two coils to be given by

$$\frac{V_1}{V_2} = \frac{N_1}{N_2}. \tag{31.11}$$

Energy conservation requires the power in the coils to be the same, that is $P_1 = P_2$. Thus, further derive

$$\frac{I_2}{I_1} = \frac{N_1}{N_2}. \tag{31.12}$$

A device operates at $V_2 = 10.0\,\mathrm{V}$. It uses a transformer to get the required voltage when plugged into a wall socket with voltage $V_1 = 120\,\mathrm{V}$. Determine the ratio of the turns in the two coils inside the transformer. (Answer: $N_1/N_2 = 12$.) If the device pulls a current of $120\,\mathrm{mA}$, determine the current coming out of the wall socket. (Answer: $I_1 = 10\,\mathrm{mA}$.)

Lecture-Example 31.7: (Induction due to change in orientation)
Consider the area enclosed by the loop formed in the configuration shown in Figure 31.4. The rotation described in the figure effectively changes the area enclosed by the loop periodically.

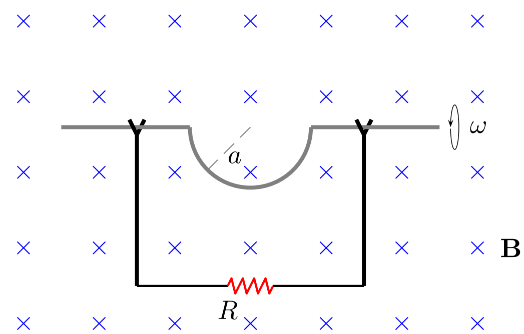

Figure 31.4: Lecture-Example 31.7

- For uniform angular speed of rotation ω we have $\theta = \omega t$ such that

  $$\frac{d\cos\theta}{dt} = -\omega \sin \omega t, \tag{31.13}$$

  show that the induced current in the loop is given by

  $$\Delta V_{\text{eff}} = BA\omega \sin \omega t. \tag{31.14}$$

  Determine the maximum induced voltage for $B = 0.1\,\mathrm{T}$, radius $a = 10\dot{c}\mathrm{m}$, and angular speed of rotation of 600 revolutions per minute ($\omega = 20\pi\,\mathrm{rad/s}$). (Answer: $0.20\,\mathrm{V}$.)

- Plot the induced voltage as a function of time.

Lecture-Example 31.8: (Generator)
A generator has a square coil consisting of 500 turns. The coil rotates at $60\,\mathrm{rad/s}$ in a $0.20\,\mathrm{T}$ magnetic field. If length of one side of the coil is $10.0\,\mathrm{cm}$, what is peak output of the generator? (Answer: $60\,\mathrm{V}$.)

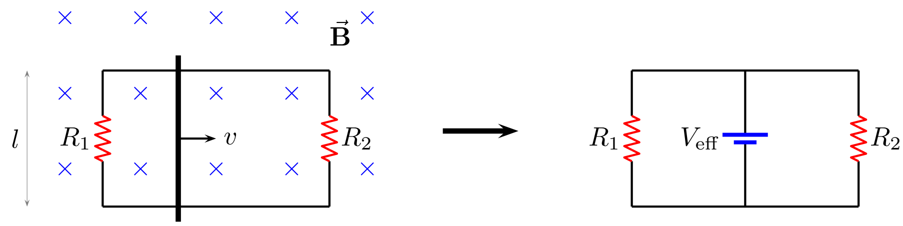

Figure 31.5: Lecture-Example 31.9

31.3 Further examples

Lecture-Example 31.9: (Kirchoff's law)
Consider the diagram shown in Figure 31.5 in which the free conducting rod is pulled with velocity v. Let $R_1 = 100.0\,\Omega$, $R_2 = 200.0\,\Omega$, $B = 0.20\,\text{T}$, $l = 10.0\,\text{cm}$, and $v = 10.0\,\text{m/s}$.

- Show that the effective voltage in the circuit (on the left) of Figure 31.5 is given by

$$V_{\text{eff}} = Blv. \tag{31.15}$$

  In particular, observe that this could be deciphered from the rate of change of flux in either of the loops. Thus, show that the effective circuit diagram on the right of Figure 31.5 is equivalent to the one on left. (Answer: $V_{\text{eff}} = 0.20\,\text{V}$.)

- Determine the currents in the two resistances. (Answer: $I_1 = 2.0\,\text{mA}$, $I_2 = 1.0\,\text{mA}$.)

Lecture-Example 31.10:
Consider the diagram shown in Figure 31.6. The rods are pulled with uniform speeds $v_1 = 10.0\,\text{m/s}$ and $v_2 = 20.0\,\text{m/s}$. Let $R_1 = 100.0\,\Omega$, $R_2 = 200.0\,\Omega$, $R_3 = 300.0\,\Omega$, $l = 10.0\,\text{cm}$, $B = 0.10\,\text{T}$.

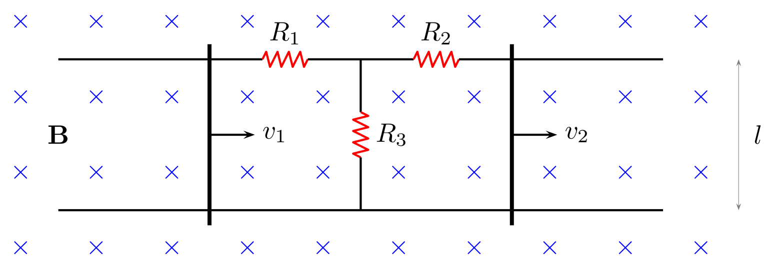

Figure 31.6: Lecture-Example 31.10

- The effective circuit is obtained by replacing each of the rods with effective voltages V_1^{eff} and V_2^{eff}. Determine the magnitude and direction sense of these effective voltages. (Answer: $V_1^{\text{eff}} = 0.10\,\text{V}$ such that the positive is touching the top rail, and $V_2^{\text{eff}} = 0.20\,\text{V}$ such the positive is touching the top rail.)

- Determine the magnitude and direction the currents in each of the resistances. (Answer: $I_1 = 0.091\,\text{mA}$ from right to left, $I_2 = 0.45\,\text{mA}$ from right to left, $I_3 = 0.36\,\text{mA}$ from top to bottom.)

Lecture-Example 31.11:
A rectangular loop of wire has an instantaneous velocity v. It is a distance y from a wire carrying current I. See Figure 31.7.

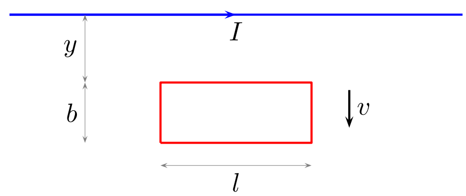

Figure 31.7: Lecture-Example 31.11

- Show that the magnetic flux Φ_B passing through the loop, at the given instant, is given by

$$\Phi_B = \frac{\mu_0 I}{2\pi} l \ln\left(1 + \frac{a}{y}\right). \tag{31.16}$$

- Show that the voltage induced in the loop, at the given instant, is given by

$$V_{\text{eff}} = \frac{\mu_0 I}{2\pi}\left(\frac{1}{y} - \frac{1}{y+a}\right) lv. \tag{31.17}$$

  Show that the magnitude of the induced current in the loop is

$$I_{\text{loop}} = \frac{V_{\text{eff}}}{R} \tag{31.18}$$

  in the clockwise direction.

- Show that the magnetic force on the loop of wire, at the given instant, is given by

$$F_{\text{loop}} = \frac{v}{R} l^2 \left[\frac{\mu_0 I}{2\pi}\left(\frac{1}{y} - \frac{1}{y+a}\right)\right]^2. \tag{31.19}$$

Chapter 32

Inductance

32.1 Inductor

An inductor is an electrical component that stores energy in the form of magnetic field. This should be contrasted with a capacitor that stores energy in the form of electric field. In both these devices the stored energy effectively creates an electric potential difference across the device. A coil of conducting wire is a simple example of an inductor. The electric potential difference across an inductor can always be expressed in the form

$$V = L\frac{dI}{dt}, \tag{32.1}$$

where L is the inductance, which is completely determined by the geometry of the inductor and the permeability of the medium that stores the magnetic energy. It is instructive to observe that the dimension of inductance is

$$[L] = [\mu_0][\text{Length}]. \tag{32.2}$$

Inductance is measured in the units of Henry, a derived unit. For calculating the (self) inductance of an arbitrary loop, or the (mutual) inductance of two current carrying wires, it is often convenient to observe that

$$L = \frac{N\Phi_B}{I}. \tag{32.3}$$

Lecture-Example 32.1: (Self inductance of a solenoid)
Consider a solenoid characterized by $n = N/l$, the number of turns N per length l, and the area of crosssection A. Show that the inductance of the solenoid is given by

$$L = \mu_0 N^2 \frac{A}{l} = \mu_0 n^2 A l. \tag{32.4}$$

- The inductance L of a solenoid scales with the volume of the solenoid. Thus, verify that the inductance per unit volume is a finite well-defined physical quantity for the solenoid.
- A solenoid of length $l = 5.0\,\text{cm}$ and radius $r = 0.50\,\text{cm}$ has $N = 1000$ turns. Determine the inductance of the solenoid. (Answer: $2.0\,\text{mH}$.)

Lecture-Example 32.2: (Inductance of a coaxial cable)
A very simple coaxial cable consists of a conducting wire sorrounded by another conductor in the shape of a right circular cylinder. More realistically, the two conductors are separated by an insulating non-magnetic

medium and also covered from outside by an insulating medium. Let us consider the case when a uniform current I flows in opposite directions in the inner and outer conductor. Let us assume the outer conductor to be infinitely thin and further presume both the conductors to be perfect conductors. Let a and b be the radius of the inner and outer conductors respectively.

- Using Ampère's law show that the magnetic field generated by the currents in the coaxial cable is

$$B(\vec{\mathbf{r}}) = \begin{cases} 0, & r < a, \\ \hat{\phi}\,\dfrac{\mu_0 I}{2\pi r}, & a < r < b, \\ 0, & b < r. \end{cases} \tag{32.5}$$

  Thus, note that the magnetic field, and the magnetic energy, is confined to regions between the two conductors.

- Evaluate the magnetic flux passing through the $\phi = 0$ plane to be

$$\Phi_B = l\frac{\mu_0 I}{2\pi}\ln\frac{b}{a}. \tag{32.6}$$

- Determine the inductance of the coaxial cable to be

$$\Phi_B = l\frac{\mu_0}{2\pi}\ln\frac{b}{a}, \tag{32.7}$$

  which depends only on the geometry of the cable.

32.2 Energy stored in an inductor

Energy consideration based on charge movement inside an inductor lets us identify the power dissipated in the inductor as

$$\frac{dU}{dt} = P = IV = IL\frac{dI}{dt} = \frac{d}{dt}\left(\frac{1}{2}LI^2\right), \tag{32.8}$$

which lets us identify the energy stored in the inductor to be

$$U_B = \frac{1}{2}LI^2, \tag{32.9}$$

up to a constant. The energy in an inductor is stored in the form of magnetic field. For the case of a coaxial cable, this statement is verified by expressing the energy per unit volume in the form

$$u_B = \frac{U_B}{\text{Volume}} = \frac{B^2}{2\mu_0}. \tag{32.10}$$

In this sense, inductance is a measure of the energy that can be lost or stored in the form of magnetic field in a region of space.

Lecture-Example 32.3: (Magnetic energy stored in a solenoid)
Using the expression for magnetic energy per unit volume of space,

$$\frac{dU_B}{dV} = \frac{B^2}{2\mu_0}, \tag{32.11}$$

and integrating over space, show that the magnetic energy stored inside a solenoid is equal to

$$U_B = \frac{1}{2} L I^2. \tag{32.12}$$

Lecture-Example 32.4: (Magnetic energy stored in a coaxial cable)
Using the expression for magnetic energy per unit volume of space,

$$\frac{dU_B}{dV} = \frac{B^2}{2\mu_0}, \tag{32.13}$$

and integrating over space, show that the magnetic energy stored inside a coaxial cable is equal to

$$U_B = \frac{1}{2} L I^2. \tag{32.14}$$

32.3 RL circuit

A resistor and inductor in series constitutes a RL circuit. An inductor resists a change in current. Thus, it is the inertia of current. An obvious scenario when sharp changes in current occur in a circuit is when the switch is turned on or off. An inductor in these instances smoothen the changes in currents.

Switching on a RL circuit

A resistor and an inductor in series with a battery is governed by the equation, using Kirchoff's law,

$$V - IR - L\frac{dI}{dt} = 0. \tag{32.15}$$

We can solve this differential equation for the initial condition

$$I(0) = 0 \tag{32.16}$$

to yield

$$I(t) = \frac{V}{R}\left[1 - e^{-\frac{t}{L/R}}\right]. \tag{32.17}$$

Thus, it takes infinite time for the current to reach its maximum value, $I(\infty) = V/R$. Nevertheless, the rate at which the current increases is governed by $\tau = L/R$, which is called the time constant of the RL circuit.

Lecture-Example 32.5: (Time constant)
Show that the current passing through a resistor at time $t = \tau = L/R$, during the process of switching on a RL circuit, is

$$I(\tau) = \frac{V}{R}\left(1 - \frac{1}{e}\right) \sim 0.632\,\frac{V}{R}. \tag{32.18}$$

- Evaluate the time constant τ for the case $R = 1.0\,\text{M}\Omega$ and $L = 1.0\,\text{mH}$. (Answer: $\tau = 1.0\,\text{ms}$.)

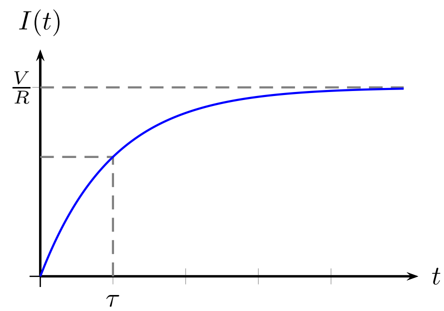

Figure 32.1: Switching on a RL circuit.

32.4 LC circuit

An inductor and a capacitor in series constitutes a LC circuit. A capacitor stores energy in the form electric field and an inductor stores energy in the form of magnetic field. Thus, an ideal LC circuit leads to oscillations in current, corresponding to the oscillations in the electric and magnetic energy.

An inductor and a capacitor in series is governed by the equation, using Kirchoff's law,

$$\frac{dI}{dt} = -\frac{Q}{LC}. \tag{32.19}$$

We can solve this differential equation for the initial conditions

$$Q(0) = Q_0, \tag{32.20a}$$
$$I(0) = 0, \tag{32.20b}$$

to yield

$$Q(t) = Q_0 \cos \omega t, \tag{32.21}$$

where the angular frequency of oscillations is given by

$$\omega = \frac{1}{\sqrt{LC}}. \tag{32.22}$$

Chapter 33

Electromagnetic waves

33.1 Maxwell's equations

Let us analyse the Ampère law for a RC circuit, while the capacitor is charging. Using Ampère's law in Fig. 33.1, and using the ambiguity in defining the surface bounded by a curve, we deduce

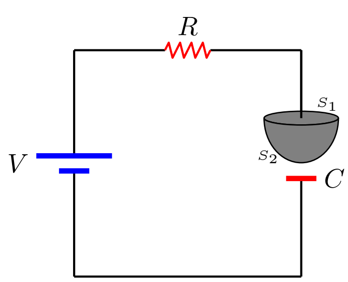

Figure 33.1: Ampère's law for an RC circuit.

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \begin{cases} \mu_0 I, & \text{for surface } S_1, \\ 0, & \text{for surface } S_2. \end{cases} \tag{33.1}$$

This apparent contradiction was removed by Maxwell by restating the Ampère law as

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}, \tag{33.2}$$

which implies that a rate of change of the electric flux can also generate a magnetic field.

Thus, the four independent laws that govern the electric and magnetic field in a region of space, in integral form, are the following.

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{\text{en}}}{\varepsilon_0} \qquad \text{(Gauss's law for } \vec{\mathbf{E}}\text{)} \tag{33.3a}$$

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0 \qquad \text{(Gauss's law for } \vec{\mathbf{B}}\text{)} \tag{33.3b}$$

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = \frac{d\Phi_B}{dt} \qquad \text{(Faraday's law)} \tag{33.3c}$$

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} + \mu_0 I \qquad \text{(Ampère's law)} \tag{33.3d}$$

The above four equations are collectively called the Maxwell equations. The symmetry in the electric and magnetic effects is striking in the above equations, which would have been complete if not for the absence of magnetic charges and magnetic currents. There is no conclusive experimental observation of magnetic charges.

The Maxwell equations in the integral form can be rewritten in differential form, using the calculus of the differential vector operator

$$\vec{\boldsymbol{\nabla}} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \tag{33.4}$$

as the following.

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = \frac{1}{\varepsilon_0}\rho \qquad \text{(Gauss's law for } \vec{\mathbf{E}}\text{)} \tag{33.5a}$$

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}} = 0 \qquad \text{(Gauss's law for } \vec{\mathbf{B}}\text{)} \tag{33.5b}$$

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = \frac{d\vec{\mathbf{B}}}{dt} \qquad \text{(Faraday's law)} \tag{33.5c}$$

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = \mu_0\varepsilon_0\frac{d\vec{\mathbf{E}}}{dt} + \mu_0\vec{\mathbf{J}} \qquad \text{(Ampère's law)} \tag{33.5d}$$

Here we introduced the charge per unit volume $\rho(\vec{\mathbf{r}})$ and the current per unit area $\vec{\mathbf{J}}(\vec{\mathbf{r}})$,

$$\int dV\,\rho(\vec{\mathbf{r}}) = Q, \tag{33.6a}$$

$$\int d\vec{\mathbf{A}} \cdot \vec{\mathbf{J}}(\vec{\mathbf{r}}) = I. \tag{33.6b}$$

33.2 Electromagnetic waves

The Maxwell equations imply that, in a region of space where there are no charges and currents, the electric and magnetic fields satisfy the wave equations

$$\nabla^2\vec{\mathbf{E}} = \frac{1}{c^2}\frac{\partial^2\vec{\mathbf{E}}}{\partial t^2}, \tag{33.7a}$$

$$\nabla^2\vec{\mathbf{B}} = \frac{1}{c^2}\frac{\partial^2\vec{\mathbf{B}}}{\partial t^2}, \tag{33.7b}$$

where the speed of these waves, called the speed of light, is

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}. \tag{33.8}$$

The meter, in SI units, is defined as the distance travelled by light in vacuum in $1/299\,792\,458$ of a second. As a consequence, the speed of light in vacuum, in SI units, is expressed as a whole number,

$$c = 299\,792\,458\,\frac{\text{m}}{\text{s}}. \tag{33.9}$$

These electromagnetic waves, which are oscillations of the electric and magnetic fields in space and time, can sustain each other.

Properties of electromagnetic waves in vacuum

1. The wave nature stipulates the relation between the wavelength λ, frequency f, and speed c of the wave,

$$c = \lambda f. \tag{33.10}$$

The time period $T = 1/f$, and the wavevector $k = 2\pi/\lambda$, are related quantities.

Frequency	Wavelength	
10^5 Hz	10^3 m	AM radio wave
10^8 Hz	10^0 m	FM radio wave
10^{11} Hz	10^{-3} m	Microwave
10^{15} Hz	10^{-6} m	Visible light
10^{17} Hz	10^{-9} m	X ray
10^{23} Hz	10^{-15} m	Gamma ray

Table 33.1: Orders of magnitude (electromagnetic wave)

2. The electromagnetic energy density is given by

$$u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2. \tag{33.11}$$

The flux of the electromagnetic energy density, a measure of the flow rate of electromagnetic energy per unit area, is given by the Poynting vector

$$\vec{\mathbf{S}} = \frac{1}{\mu_0} \vec{\mathbf{E}} \times \vec{\mathbf{B}}. \tag{33.12}$$

The electromagnetic momentum density is given by

$$\vec{\mathbf{G}} = \frac{1}{c^2} \vec{\mathbf{S}}. \tag{33.13}$$

3. The Maxwell equations constraint the directions of the electric field, the magnetic field, and the direction of propagation to be mutually perpendicular,

$$\vec{\mathbf{E}} \times \vec{\mathbf{B}} = \hat{\mathbf{k}}\, c\mu_0 u, \qquad \vec{\mathbf{E}} \cdot \vec{\mathbf{B}} = 0. \tag{33.14}$$

Further, we have

$$E = cB. \tag{33.15}$$

Lecture-Example 33.1: (Absorption coefficient of liquid water)
Using the absorption coefficient of water as a function of frequency presented in the following link

http://www.britannica.com/science/absorption-coefficient/images-videos

argue that kilometer long waves (extremely low-frequency waves) are suitable candidates for communications between land base and submarines. (Inverse of absorption coefficient is a measure of how deep the wave will travel in water before getting absorbed.)

Lecture-Example 33.2: (X-ray telescope)
Using the opacity of electromagnetic waves as a function of the wavelength of eleactromagnetic waves presented in the following link

Wikipedia: Opacity of atmosphere to electromagnetic waves

argue that an X-ray telescope has to be necessarily installed, above the atmosphere, in space. Further, discuss radio-wave astronomy and gamma-ray astronomy.

Part III

Optics

Chapter 34

Ray Optics: Reflection

Visible light is an electromagnetic wave, oscillations of electric and magnetic fields in space and time. The wavelength of visible light is in the range of $0.400-0.700\,\mu$m. When the size of irregularities at the interface of two mediums is smaller than the wavelength of visible light, it is a good approximation to treat the electromagnetic wave by rays of light, along the direction of propagation of the waves, which are perpendicular to the surfaces formed by the wave fronts. The study of propagation of light, in this straight line approximation, is called ray optics.

Visible light, and other electromagnetic waves, can not penetrate into a perfect conductor, because electric field has to be zero inside a perfect conductor. A metal, like gold and silver, is a perfect conductor to a good approximation. The surface of a metal is naturally smooth. The surface of a perfect conductor will be called a mirror. The mirrors we typically find in daily use, consists of a slab of glass with a coating of metal on one of the surfaces of the slab.

34.1 Law of reflection

Propagation of light at the interface of a medium and a mirror is governed by the law of reflection that states that the angle of incidence is equal to the angle of reflection.

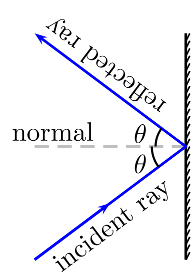

Figure 34.1: A ray of light reflected by a mirror.

Image formation as a perception of our eye

Our eye extrapolates two or more rays of light and the point of intersection of these rays is perceived as a source or image. If the light passes through the point of intersection of the extrapolated rays, it is perceived as a real image. Image formed by a overhead projector is a real image. If the light does not pass through the point of intersection of the extrapolated rays, it is perceived as a virtual image. Image formed by a bathroom mirror is a virtual image.

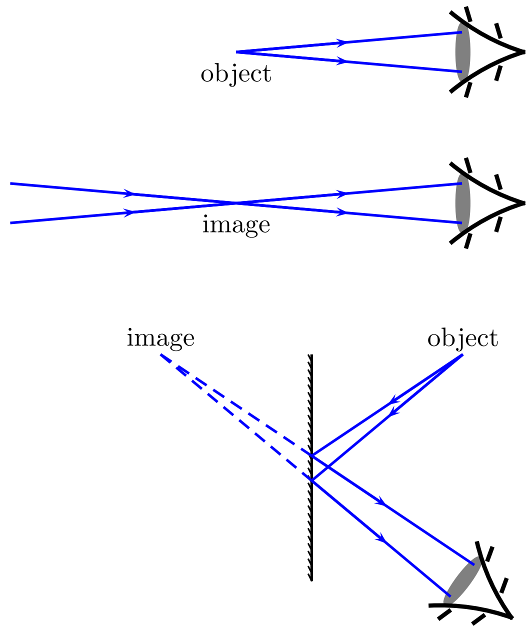

Figure 34.2: Images as perceived by an eye.

Lecture-Example 34.1: (Optimal mirror placement)
Your height is h. The vertical distance between your eye and top of head is h_1, and between your eye and toe is h_2.

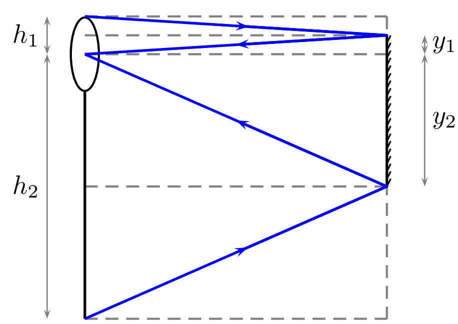

Figure 34.3: Lecture-Example 34.1

- What is the minimum height $y = y_1 + y_2$ of a mirror you need to place on a vertical wall in which you can see your complete image?
- Does your answer depend on how far away you stand from the mirror?

Lecture-Example 34.2:
Given $\alpha = 30.0°$, in Figure 34.4. Show that $\theta = 2\alpha$.

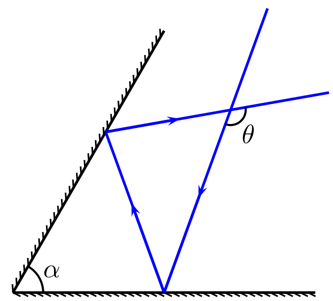

Figure 34.4: Lecture-Example 34.2

34.2 Spherical mirrors

When the surface of a mirror is part of a sphere, it is called a spehrical mirror. If the inner side of the part of sphere forms the reflecting surface, it is called a concave mirror. If the outer side of the part of sphere forms the reflecting surface, it is called a convex mirror. The center of the sphere, of which the mirror is a part, is called the center of curvature. The radius of this sphere is called the radius of curvature. A line passing through the center of curvature and the center of the mirror will be defined to the optical axis, the direction being that of a light ray. The focal point is the point half way between the center of curvature and center of mirror, and the corresponding distance is the focal length,

$$f = \frac{R}{2}. \tag{34.1}$$

The sign conventions, and the related terminologies, is summarized in Figure 34.5.

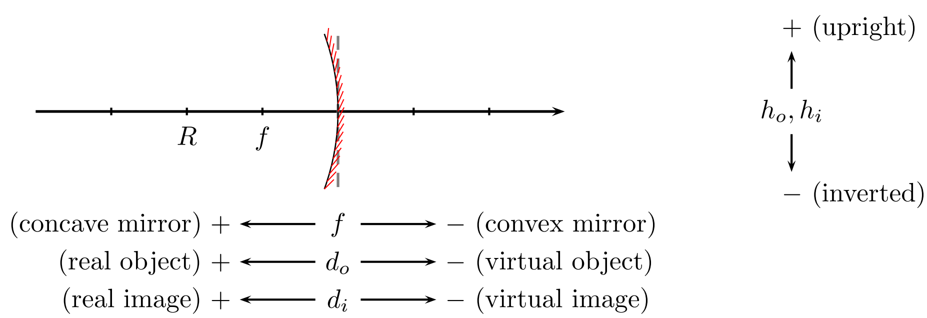

Figure 34.5: Sign conventions for spherical mirrors. The mirror pictured is a concave mirror.

Mirror formula

Using the law of reflection and the geometry of a circle we can deduce the mirror formula

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}, \tag{34.2}$$

and the expression for magnification,

$$m = \frac{h_i}{h_o} = -\frac{d_i}{d_o}. \tag{34.3}$$

Lecture-Example 34.3: (Plane mirror)
A plane mirror has an infinite radius of curvature. Using the mirror formula, conclude that the image distance is equal to the negative of the object distance, $d_i = -d_o$. Thus, deduce that, the image formed when you stand in front of a plane mirror is virtual and upright.

Lecture-Example 34.4: (Concave mirror)
An object of height $h_o = 1.0\,\text{cm}$ is placed upright at a distance d_o in front of a concave mirror. The mirror's focal length is $f = +10.0\,\text{cm}$.

- Let $d_o = +30.0\,\text{cm}$. Calculate the image distance. (Answer: $d_i = +15\,\text{cm}$.) What is the magnification? (Answer: $m = -0.50$.) Is the image real or virtual? Is the image inverted or upright? Verify your results using a ray diagram drawn.
- Repeat for $d_o = +20.0\,\text{cm}$. (Answer: $d_i = +20.0\,\text{cm}$, $m = -1.0$.)
- Repeat for $d_o = +15.0\,\text{cm}$. (Answer: $d_i = +30.0\,\text{cm}$, $m = -2.0$.)
- Repeat for $d_o = +10.0\,\text{cm}$. (Answer: $d_i \to \pm\infty\,\text{cm}$, $m \to \pm\infty$.)
- Repeat for $d_o = +5.0\,\text{cm}$. (Answer: $d_i = -10.0\,\text{cm}$, $m = +2.0$.)

Lecture-Example 34.5: (Convex mirror)
An object of height $h_o = 1.0\,\text{cm}$ is placed upright at a distance d_o in front of a convex mirror. The mirror's focal length is $f = -10.0\,\text{cm}$.

- Let $d_o = +30.0\,\text{cm}$. Calculate the image distance. (Answer: $d_i = -7.5\,\text{cm}$.) What is the magnification? (Answer: $m = +0.25$.) Is the image real or virtual? Is the image inverted or upright? Verify your results using a ray diagram drawn.
- Verify that the image is always virtual, upright, and diminished.
- Rear view mirrors on automobiles are convex mirrors. Understand the following warning statement regarding rear viw mirrors, “Objects in mirror are closer than they appear”.

Chapter 35

Ray optics: Refraction

35.1 Index of refraction

Electromagnetic waves travel at the speed of light c in vacuum. But, they slow down in a medium. The refractive index of a medium

$$n = \frac{c}{v} \tag{35.1}$$

is a measure of the speed of light v in the medium. Refractive index of a medium is always greater than or equal to unity. The speed of light in a medium varies with the color of light. Thus, for the same medium, the refractive index changes with the color of light, a phenomena called dispersion.

35.2 Law of refraction

The law of refraction, or Snell's law, relates the angle of incidence and angle of refraction at the interface of two mediums,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{35.2}$$

It can be derived using Fermat's principle that states that light takes the path of least time. As a consquence, a ray light bends towards the normal when it goes from a denser to a rarer medium, and vice versa.

Lecture-Example 35.1: (Apparent depth)
Determine the apparent depth h' of a swimming pool of real depth h.

- Show that

$$h' \tan\theta_1 = h \tan\theta_2. \tag{35.3}$$

  Then, show that, for small angles we have

$$h' = \frac{n_1}{n_2} h. \tag{35.4}$$

  Evaluate the apparent depth for $h = 2.0\,\text{m}$, $n_1 = 1.0$, and $n_2 = 1.33$. (Answer: $h' = 1.5\,\text{m}$.)

1	vacuum
1.0003	air
1.33	water
1.5	glass
2.4	diamond

Table 35.1: Orders of magnitude (refractive index).

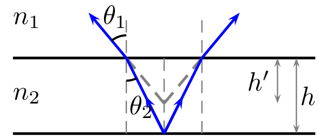

Figure 35.1: Lecture-Example 35.1

Lecture-Example 35.2: (Prism)
Light travels through a prism made of glass ($n = 1.5$) as shown in Figure 35.2. Given $\alpha = 50°$ and $i_1 = 45°$. Determine the angle of deviation δ.

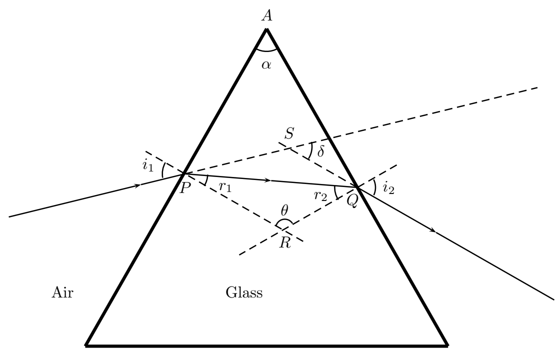

Figure 35.2: Lecture-Example 35.2

35.3 Total internal reflection

When light passes from a denser to a rarer medium, it bends away from the normal. As a consequence, there exists a critical angle beyond which there is no refraction. The critical angle is determined using $\theta_2 = 90°$, for $n_1 > n_2$,

$$n_1 \sin\theta_c = n_2. \tag{35.5}$$

Lecture-Example 35.3:
The index of refraction of benzene is 1.80. Determine the critical angle for total internal reflection at a benzene-air interface. (Answer: $\theta_c = 33.8°$.)

Lecture-Example 35.4: (Fiber optic cable)
Figure 35.3 shows the crosssection of a fiber optic cable made out of a material of refractive index n in air. Light needs to satisfy the conditions for total internal reflection at the interfaces to avoid loss, which in turn defines an acceptance cone of (total) angle 2θ.

- Using Snell's law at the interface (of the feeding side) show that

$$1.0 \sin\theta = n \sin(90° - \theta_c). \tag{35.6}$$

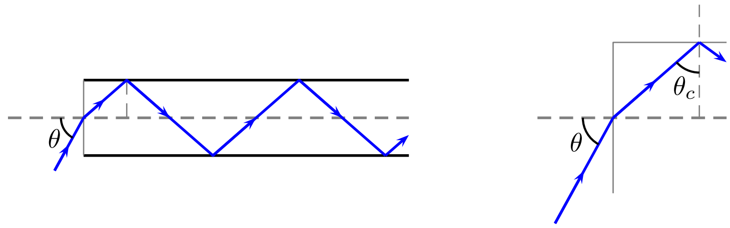

Figure 35.3: Lecture-Example 35.4

Using Snell's law at the interface (forming the circular side of cylinder) show that

$$n \sin\theta_c = 1.0 \sin 90°. \tag{35.7}$$

Show that the (half) acceptance angle θ satisfies the relation

$$\sin\theta \leq \sqrt{n^2 - 1}. \tag{35.8}$$

- Evaluate the (half) acceptance angle for water ($n = 1.33$). This allows light to follow a stream of water, giving the impression of bending of light. For example, check out this YouTube video.
- Evaluate the (half) acceptance angle for a material medium with refractive index $n = \sqrt{2}$. What happens when the material medium has a refractive index larger than $\sqrt{2}$? For example, glass with $n = 1.5$.

Lecture-Example 35.5: (Examples)

- Optical phenomenon called mirage.

35.4 Thin spherical lens

When the surface of the interfaces enclosing a medium is spherical in shape, on both sides, it is called a thin spherical lens. The focal length of a thin spherical lens is given in terms of the radius of curvatures of the two interfaces, R_1 and R_2,

$$\frac{1}{f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_1}\right]. \tag{35.9}$$

The sign conventions, and the related terminologies, is summarized in Figure 35.4.

Lens formula

Using the law of refraction and the geometry of a circle we can deduce the lens formula

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}, \tag{35.10}$$

and the expression for magnification,

$$m = \frac{h_i}{h_o} = -\frac{d_i}{d_o}. \tag{35.11}$$

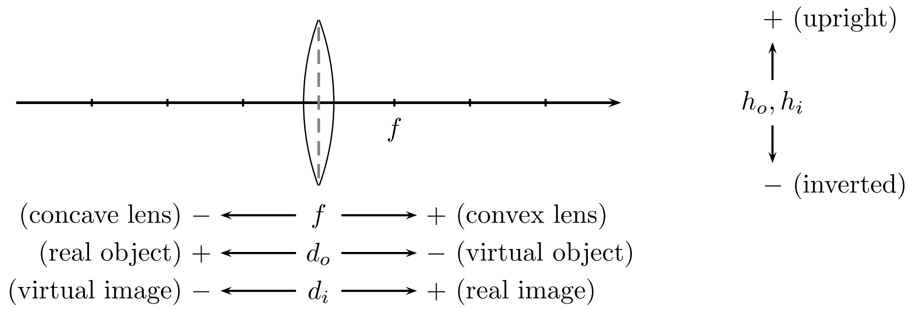

Figure 35.4: Sign conventions for spherical lenses.

Lecture-Example 35.6: (Convex lens)
An object of height $h_o = 1.0\,\text{cm}$ is placed upright at a distance d_o in front of a convex lens. The lens' focal length is $f = +10.0\,\text{cm}$.

- Let $d_o = +30.0\,\text{cm}$. Calculate the image distance. (Answer: $d_i = +15\,\text{cm}$.) What is the magnification? (Answer: $m = -0.50$.) Is the image real or virtual? Is the image inverted or upright? Verify your results using a ray diagram drawn.
- Repeat for $d_o = +20.0\,\text{cm}$. (Answer: $d_i = +20.0\,\text{cm}$, $m = -1.0$.)
- Repeat for $d_o = +15.0\,\text{cm}$. (Answer: $d_i = +30.0\,\text{cm}$, $m = -2.0$.)
- Repeat for $d_o = +10.0\,\text{cm}$. (Answer: $d_i \to \pm\infty\,\text{cm}$, $m \to \pm\infty$.)
- Repeat for $d_o = +5.0\,\text{cm}$. (Answer: $d_i = -10.0\,\text{cm}$, $m = +2.0$.)

Lecture-Example 35.7: (Concave lens)
An object of height $h_o = 1.0\,\text{cm}$ is placed upright at a distance d_o in front of a concave lens. The lens' focal length is $f = -10.0\,\text{cm}$.

- Let $d_o = +30.0\,\text{cm}$. Calculate the image distance. (Answer: $d_i = -7.5\,\text{cm}$.) What is the magnification? (Answer: $m = +0.25$.) Is the image real or virtual? Is the image inverted or upright? Verify your results using a ray diagram drawn.
- Verify that the image is always virtual, upright, and diminished.