Midterm Exam No. 02 (2021 Spring)

PHYS 420: Electricity and Magnetism II

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1. (**20 points.**) Magnets are described by their magnetic moment. Estimate the magnetic moment $\mathbf{m}$ of Earth, assuming it to be a point magnetic dipole. Assume the magnitude of the Earth's magnetic field on its surface at the North pole to be $0.7 \times 10^{-4}\,\mathrm{T} = 0.7\,\mathrm{Gauss}$. Show your work.

2. (**20 points.**) A typical bar magnet is suitably approximated as a point magnetic dipole moment $\mathbf{m}$. The magnetic field due to a point magnetic dipole $\mathbf{m}$ at a distance $\mathbf{r}$ away from the magnetic dipole is given by the expression

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\big[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\big]}{r^3}, \qquad r \neq 0. \tag{1}$$

   Consider the case when the point dipole is positioned at the origin and is pointing in the z-direction, i.e., $\mathbf{m} = m\,\hat{\mathbf{z}}$.

   (a) Qualitatively (and neatly) plot the magnetic field lines for the dipole $\mathbf{m}$. (Hint: You do not have to depend on Eq. (1) for this purpose. An intuitive knowledge of magnetic field lines should be the guide.)

   (b) Find the expression for the magnetic field on the negative x-axis.

3. (**20 points.**) Determine the total magnetic dipole moment for the following configuration. The current in the loop is I and each fold in the loop is of length a.

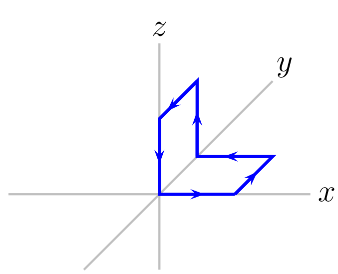

Figure 1: Problem 3

4. (**20 points.**) A four-vector in the context of Lorentz tranformation can be described using the notation

$$a^\alpha = (a^0, a^1, a^2, a^3). \tag{2}$$

Let

$$b^\alpha = (b^0, b^1, b^2, b^3) \tag{3}$$

be another four-vector. The scalar product between two Lorentz vectors is given by

$$a^\alpha b_\alpha = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3. \tag{4}$$

The square of the 'length' of the four-vector a^α is given by

$$a^\alpha a_\alpha, \tag{5}$$

which is not necessarily positive. The length of a four-vector is invariant, that is, it is independent of the Lorentz frame. If two Lorentz four-vectors are orthogonal they satisfy

$$a^\alpha b_\alpha = 0. \tag{6}$$

Orthogonality is an invariant concept.

(a) Determine the length of

$$p^\alpha = (5, 0, 0, 3), \tag{7}$$

where the numbers are in arbitrary units. Is it time-like, light-like, or space-like?

(b) Find a four-vector of the form

$$q^\alpha = (q^0, 0, 0, q^3) \tag{8}$$

that is perpendicular to p^α.

5. (**20 points.**) This problem is mostly a reading exercise. Go through the following discussion. The additional term $3\pi k^4/64$ in Eq. (13a) below is the higher order term necessary to fix the discrepancy encountered in the class. Then, evaluate the magnetic vector potential and the magnetic field on the axis of the loop. This is achieved by letting $\rho \to 0$ in the last two equations below.

A circular loop of radius a carrying a steady current I with the loop chosen to be in the x-y plane with the origin at the center of the loop has the the magnetic vector potential given by

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{4\pi} \frac{4a}{\sqrt{z^2 + (\rho + a)^2}} \left[\frac{2}{k^2} \Big\{ K(k) - E(k) \Big\} - K(k) \right], \tag{9}$$

where

$$k^2 = \frac{4a\rho}{z^2 + (\rho + a)^2}. \tag{10}$$

The magnetic field is

$$\mathbf{B}(\mathbf{r}) = \hat{\mathbf{z}}\frac{\mu_0 I}{4\pi}\frac{2}{\sqrt{z^2+(\rho+a)^2}}\left[K(k) - \frac{(z^2+\rho^2-a^2)}{z^2+(\rho-a)^2}E(k)\right]$$
$$-\hat{\boldsymbol{\rho}}\frac{\mu_0 I}{4\pi}\frac{2}{\sqrt{z^2+(\rho+a)^2}}\frac{z}{\rho}\left[K(k) - \frac{(z^2+\rho^2+a^2)}{z^2+(\rho-a)^2}E(k)\right]. \tag{11}$$

We can evaluate the vector potential and the magnetic field close to the symmetry axis of the loop using the approximation $k^2 \ll 1$ in the above expressions. Using

$$\frac{(z^2+\rho^2+a^2)}{z^2+(\rho-a)^2} = \frac{(2-k^2)}{2(1-k^2)} = 1 + \frac{k^2}{2} + \frac{k^4}{2} + \ldots, \tag{12a}$$

$$\frac{(z^2+\rho^2-a^2)}{z^2+(\rho-a)^2} = \frac{(2-k^2)}{2(1-k^2)} - \frac{a}{2\rho}\frac{k^2}{(1-k^2)}$$
$$= \left[1 + \frac{k^2}{2} + \frac{k^4}{2} + \ldots\right] - \frac{a}{2\rho}\left[k^2 + k^4 + \ldots\right] \tag{12b}$$

we can show that

$$\frac{2}{k^2}\Big\{K(k) - E(k)\Big\} - K(k) = \frac{\pi}{16}k^2 + \frac{3\pi}{64}k^4 + \ldots \tag{13a}$$
$$= \frac{(\pi a^2)}{[z^2+(\rho+a)^2]}\frac{\rho}{4a}\left[1 + \frac{3a\rho}{[z^2+(\rho+a)^2]} + \ldots\right], \tag{13b}$$

$$K(k) - \frac{(z^2+\rho^2-a^2)}{z^2+(\rho-a)^2}E(k) = \left[-\frac{3\pi}{32}k^4 + \ldots\right] + \frac{\pi a}{4\rho}\left[k^2 + \frac{3}{4}k^4 + \ldots\right]$$
$$= \frac{(\pi a^2)}{[z^2+(\rho+a)^2]}\left[1 - \frac{3}{2}\frac{\rho(\rho-2a)}{[z^2+(\rho+a)^2]} + \ldots\right], \tag{13c}$$

$$K(k) - \frac{(z^2+\rho^2+a^2)}{z^2+(\rho-a)^2}E(k) = -\frac{3\pi}{32}k^4 + \ldots$$
$$= -\frac{3}{2}\frac{(\pi a^2)\rho^2}{[z^2+(\rho+a)^2]^2} + \ldots. \tag{13d}$$

Using these approximations, which are appropriate for regions close to the axis ($k^2 \ll 1$,) we have

$$\mathbf{A}(\mathbf{r}) \xrightarrow{k^2 \ll 1} \hat{\boldsymbol{\phi}}\,\frac{\mu_0}{4\pi}\frac{I(\pi a^2)\rho}{[z^2+(\rho+a)^2]^{\frac{3}{2}}}\left[1 + \frac{3a\rho}{[z^2+(\rho+a)^2]}\right] \tag{14}$$

and

$$\mathbf{B}(\mathbf{r}) \xrightarrow{k^2 \ll 1} \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{I(\pi a^2)2}{[z^2+(\rho+a)^2]^{\frac{3}{2}}}\left[1 - \frac{3}{2}\frac{\rho(\rho-2a)}{[z^2+(\rho+a)^2]}\right] - \hat{\boldsymbol{\rho}}\frac{\mu_0}{4\pi}\frac{I(\pi a^2)3\rho z}{[z^2+(\rho+a)^2]^{\frac{5}{2}}}. \tag{15}$$