Homework No. 03 (2021 Spring)

PHYS 420: ELECTRICITY AND MAGNETISM II

Department of Physics, Southern Illinois University–Carbondale

Due date: Friday, 2021 Feb 12, 2:00 PM

0. Keywords: Magnetostatics (Chap. 5, Griffiths 4th edition), Magnetic vector potential, Biot-Savart law (Sec. 5.2, Griffiths 4th edition).

0. Problems 2 and 4 are to be submitted for assessment. Rest are for practice.

1. (**20 points.**) The magnetic field $\mathbf{B}(\mathbf{r})$ is given in terms of the magnetic vector potential $\mathbf{A}(\mathbf{r})$ by the relation
$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{1}$$
Find a magnetic vector potential (up to a gauge) for the uniform magnetic field
$$\mathbf{B} = B\,\hat{\mathbf{z}}. \tag{2}$$
Then, find another solution for $\mathbf{A}$ (up to a gauge) that is different from your original solution by more than just a constant. If you designed an experiment to measure $\mathbf{A}$, which one of your solution will the experiment measure?

2. (**20 points.**) A homogeneous magnetic field $\mathbf{B}$ is characterized by the vector potential
$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}. \tag{3}$$
   (a) Evaluate $\boldsymbol{\nabla} \times \mathbf{A}$.
   (b) Verify that this construction satisfies the radiation gauge by showing that
$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0. \tag{4}$$
   (c) Is this construction unique?

3. (**20 points.**) Is it correct to conclude that
$$\boldsymbol{\nabla} \cdot (\mathbf{r} \times \mathbf{A}) = -\mathbf{r} \cdot (\boldsymbol{\nabla} \times \mathbf{A}), \tag{5}$$
where $\mathbf{A}$ is a vector dependent on $\mathbf{r}$? Explain your reasoning.

4. (**50 points.**) (Based on Problem 5.8, Griffiths 4th edition.)
The magnetic field at position $\mathbf{r} = (x, y, z)$ due to a finite wire segment of length $2L$

carrying a steady current I, with the caveat that it is unrealistic (why?), placed on the z-axis with its end points at $(0, 0, L)$ and $(0, 0, -L)$, is

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{4\pi}\frac{1}{\sqrt{x^2+y^2}}\left[\frac{z+L}{\sqrt{x^2+y^2+(z+L)^2}} - \frac{z-L}{\sqrt{x^2+y^2+(z-L)^2}}\right], \tag{6}$$

where $\hat{\boldsymbol{\phi}} = (-\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}) = (-y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}})/\sqrt{x^2+y^2}$.

(a) Show that by taking the limit $L \to \infty$ we obtain the magnetic field near a long straight wire carrying a steady current I,

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{2\pi\rho}, \tag{7}$$

where $\rho = \sqrt{x^2+y^2}$ is the perpendicular distance from the wire.

(b) Show that the magnetic field on a line bisecting the wire segment is given by

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{2\pi\rho}\frac{L}{\sqrt{\rho^2+L^2}}. \tag{8}$$

(c) Find the magnetic field at the center of a square loop, which carries a steady current I. Let $2L$ be the length of a side, ρ be the distance from center to side, and $R = \sqrt{\rho^2+L^2}$ be the distance from center to a corner. (Caution: Notation differs from Griffiths.) You should obtain

$$B = \frac{\mu_0 I}{2R}\frac{4}{\pi}\tan\frac{\pi}{4}. \tag{9}$$

(d) Show that the magnetic field at the center of a regular n-sided polygon, carrying a steady current I is

$$B = \frac{\mu_0 I}{2R}\frac{n}{\pi}\tan\frac{\pi}{n}, \tag{10}$$

where R is the distance from center to a corner of the polygon.

(e) Show that the magnetic field at the center of a circular loop of radius R,

$$B = \frac{\mu_0 I}{2R}, \tag{11}$$

is obtained in the limit $n \to \infty$.

5. (**20 points.**) The vector potential for a straight wire of infinite extent carrying a steady current I is

$$\mathbf{A}(\mathbf{r}) = \hat{\mathbf{z}}\,\frac{\mu_0 I}{2\pi}\ln\frac{2L}{\rho}, \tag{12}$$

with $L \to \infty$ understood in the equation. The magnetic field around the wire is given by

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{2\pi\rho}. \tag{13}$$

(a) Draw the field lines for the above vector potential and the magnetic field. Be precise.

(b) Evaluate $\boldsymbol{\nabla} \times \mathbf{A}$.

6. (**20 points.**) The magnetic field for a straight wire of infinite extent carrying a steady current I is given by

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi\rho}. \tag{14}$$

Verify that

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{15}$$

everywhere. In particular, investigate if the magnetic field is divergenceless on the wire, where $\rho = 0$. Next, evaluate

$$\boldsymbol{\nabla} \times \mathbf{B} \tag{16}$$

everywhere. Thus, check if the magnetic field due to a straight current carrying wire satisfies the two Maxwell equations relevant for magnetostatics.