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On the field of equal charges at the corners of a regular polygon

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I. THREE EQUAL CHARGES AT THE CORNERS OF AN EQUILATERAL TRIANGLE

A problem that appears in at least two current textbooks on elementary physics[1] goes something like this:

> Three equal charges are arranged on the corners of an equilateral triangle. Where does the value of the electrostatic field equal zero?

Practically all my colleagues and my students (as well as the answer keys supplied with the cited texts) reply to this question with something like this: "At the center of the triangle, by symmetry." If I ask "Where else is the field equal to zero?" there is usually a short pause and my colleagues and students say "At infinity." But then when I ask "And where *else*?" the pause is longer.

If I suggest that there are three other points at which the electric field is zero, a common response is "Oh, really?"—expressing a combination of doubt and surprise. It is amusing that such an apparently simple problem surprises us with its answer. Where are these other zeros?

Certainly the other zeros of the electric field will lie on symmetry axes in the plane of the charges. Let us place the center of the triangle at the origin and the three charges, each $+q$, in the x–y plane at $(-a,0)$, $(a/2,\sqrt{3}a/2)$, and $(a/2,-\sqrt{3}a/2)$. On the x axis, which is an axis of symmetry, we calculate the electric field directly for $x > -a$:

$$E_x = \frac{q}{4\pi\epsilon_0}\left(\frac{2(x-a/2)}{[(x-a/2)^2+3a^2/4]^{3/2}} + \frac{1}{(x+a)^2}\right). \quad (1)$$

As expected, $E_x(0) = 0$. But if we calculate the derivative dE_x/dx at the origin, we find it to be nonzero and negative. Now, close to the charge on the x axis $E_x \to +\infty$, and $E_x > 0$ as $x \to +\infty$. Thus, a negative value of dE_x/dx at the origin implies, as a simple sketch of E_x vs x will show, that there is at least one more point with $E_x = 0$ for which $x > 0$. In fact, setting $E_x = 0$ and solving for x in Eq. (1) we find that this point is at $x = 0.284718a$, or a distance from the center that is 0.164382 times the length of one side of the triangle.

From the threefold symmetry of the charges, we know there are two more off-center points where the field is zero.

After working out this problem I found that the problem of three charges had been discussed quite completely by Sir James Jeans[2] in his textbook on electricity and magnetism. For reasons I do not understand, he gives the location of the field zero at a value of x "just less than $\frac{1}{4}a$" (instead of $0.285a$), he calculates the potential at this field zero to be $3.04\ q/4\pi\epsilon_0 a$ (instead of the correct value $3.02\ q/4\pi\epsilon_0 a$), and he sketches the equipotential line through the field zeros incorrectly (it does not cross itself at right angles in the plane of the charges). Figure 1 shows an improved sketch of several electric field and equipotential lines in the plane of the charges.

Then, after circulating a draft of this note, my colleague William Madigan showed me a treatment of the problem in a very completely illustrated monograph on electrostatics by E. Durand[3] in which field lines and equipotentials are plotted for a large number of charge distributions. The problem of four equal charges on the corners of a square is presented in accurate detail by Durand, and the problem of three equal charges on the corners of an equilateral triangle is left as an exercise (although an answer to the exercise is given with a somewhat inaccurate sketch of the equipotential and electric field lines).

II. EQUAL CHARGES AT THE CORNERS OF ANY REGULAR POLYGON

In the electric field of equal charges placed at the corners of an N-sided regular polygon, there are N noncentral zeros. A straightforward way to show this fact is as follows. Let the polygon be inscribed in a circle of radius a. For the region $r < a$ the electrostatic potential may be expanded[4]:

$$\Phi(\mathbf{r}) = \sum_n \frac{q_n}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}_n|} = \sum_{l,m} \frac{Q_{lm} r^l Y_{lm}(\theta,\phi)}{\epsilon_0(2l+1)},$$

where

$$Q_{lm} = \sum_n \frac{q_n Y^*_{lm}(\theta_n,\phi_n)}{r_n^{l+1}},$$

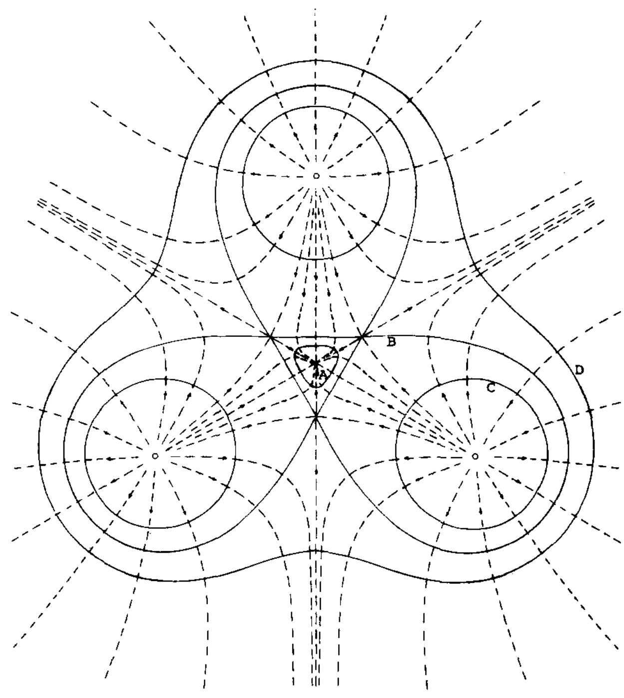

Fig. 1. Several equipotential lines (solid) and electric field lines (dashed) in the plane of three equal charges, each $+q$, at the corners of an equilateral triangle. The equipotentials shown are for $\Phi = 3.0076$ (A), 3.0196 (B), 3.7040 (C), and 2.5080 (D), in units of $q/4\pi\epsilon_0 a$, where a is the radius of the circle on which the charges lie. The electric field is zero at the center and at the crossings of B.

$\mathbf{r} = (r,\theta,\phi)$ is the field point, and $\mathbf{r}_n = (r_n,\theta_n,\phi_n)$ is the position of the nth charge q_n.

For N regularly spaced equal charges q on a circle of radius a whose plane is perpendicular to the polar axis, $\theta_n = \pi/2$,

$$Q_{00} = Nq/a\sqrt{4\pi}\,,$$

$$Q_{20} = -\sqrt{5/4\pi}Nq/2a^3\,,$$

$$Q_{10} = Q_{11} = Q_{21} = 0,$$

and

$$Q_{22} = 0 \text{ (for } N \geqslant 3).$$

In the plane of the charges the potential then becomes

$$\Phi\left(r,\frac{\pi}{2},\phi\right) = \frac{Nq}{4\pi\epsilon_0 a}\left(1 + \frac{r^2}{4a^2} + \cdots\right). \qquad (2)$$

Since $E_r = -\partial\Phi/\partial r$, for $r \ll a$ we see that $E_r < 0$. On each of the N lines from the origin through the midpoint between two adjacent charges, the value of E_r is continuous, and for $r \gg a$, $E_r \cong Nq/4\pi\epsilon_0 r^2$. Thus there must be a point on each line where E_r changes sign and $\mathbf{E} = 0$. As N increases, this point gets closer to the circle on which the charges lie.

Another way to see the result in Eq. (2) is as follows. On the z axis, perpendicular to the plane of the charges,

$$\Phi(z) = \frac{Nq}{4\pi\epsilon_0 a}\left(1 - \frac{z^2}{2a^2} + \cdots\right).$$

Since there is no charge at the origin, $\nabla^2\Phi = 0$ requires that Φ must also have a quadratic dependence on the distance from the z axis. This implies that, close to the origin, the equipotential surfaces intersect the plane of the charges in concentric ellipses or hyperbolas. But for three or more charges, the only such curves which have the appropriate rotational symmetry are circles. Thus, close to the origin, the potential is cylindrically symmetric (which explains why $Q_{22} = Q_{21} = 0$), and

$$\Phi(x,y,z) = \frac{Nq}{4\pi\epsilon_0 a}\left(1 - \frac{z^2}{2a^2} + A(x^2+y^2) + \cdots\right).$$

To evaluate the constant A, we note that $\nabla^2\Phi = \partial^2 E_x/\partial x^2 + \partial^2 E_y/\partial y^2 + \partial^2 E_z/\partial z^2 = 0$ which requires that $A = 1/4a^2$, and we have again obtained the result shown in Eq. (2).

[1]R. T. Weidner and R. L. Sells, *Elementary Classical Physics* (Allyn and Bacon, Boston, 1973), 2nd ed., p. 465. R. A. Serway, *Physics for Scientists and Engineers* (Saunders, Philadelphia, 1983), p. 422.

[2]Sir James Jeans, *The Mathematical Theory of Electricity and Magnetism* (Cambridge University Press, London, 1933), 5th ed., pp. 54–56.

[3]E. Durand, *Electrostatique, Tome I:Les Distributions* (Masson, Paris, 1964).

[4]See, for example, J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), 2nd ed.

SOLUTION TO PROBLEM ON PAGE 247

Since the only force is the gravitational force, which is conservative, mechanical energy is conserved. Picking the zero of potential energy at the ground, the initial energy is entirely kinetic energy.

$$E_i = \tfrac{1}{2}mv_0^2 = \tfrac{1}{2}m(v_{x0}^2 + v_{y0}^2)$$

by the Pythagorean theorem. At the window, the ball is moving horizontally, and since there is no acceleration in the horizontal direction

$$v_y = 0 \text{ and } v_x = v_{x0}.$$

The final energy is therefore

$$E_f = mgh + \tfrac{1}{2}mv_{x0}^2,$$

$E_i = E_f$ by energy conservation,

$$\tfrac{1}{2}mv_{x0}^2 + \tfrac{1}{2}mv_{y0}^2 = mgh + \tfrac{1}{2}mv_{x0}^2,$$

$$v_{y0} = \sqrt{2gh}\,.$$

This illustrates the physics involved that the height h determines the initial vertical component of the velocity.

The next question of physics which needs to be answered is, "What determines the initial horizontal component of the velocity?" The answer is the distance D and the time t; the distance is given but not the time. At this point students should recognize that the time for the projectile to reach its maximum height is determined by the initial vertical component of the velocity; so back to the vertical motion.

$a_y = (v_y - v_{y0})/t$ from the definition of acceleration (constant a),

$$-g = (0 - v_{y0})/t,$$

$$t = v_{y0}/g = \sqrt{2gh}/g = \sqrt{2h/g}.$$

From t (and D) we can now determine v_{x0}, realizing that v_x is constant.

$$v_x = v_{x0} = D/t = D\sqrt{g/2h}\,.$$

Thus we have found $v_{y0} = \sqrt{2gh}$ and $v_{x0} = D\sqrt{g/2h}$, which uniquely specify the initial velocity in terms of its components. These can be converted to the alternate representation of velocity, namely speed and angle, in the standard way, giving the answers of Hudson.

$$v_0 = \sqrt{v_{x0}^2 + v_{y0}^2} = \sqrt{gD^2/2h + 2gh}$$

$$= \sqrt{g(D^2 + 4h^2)/2h}\,,$$

$$\tan\theta = v_{y0}/v_{x0} = \sqrt{2gh}/D\sqrt{g/2h} = 2h/D.$$

I prefer the solution in terms of v_{x0} and v_{y0} since it shows explicitly that v_{y0} is determined by h alone and that v_{x0} depends on both h and D.

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