# (Preview of) Final Exam (2022 Spring)

## PHYS 520B: ELECTROMAGNETIC THEORY

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1. (**20 points.**) Not available in preview mode.
2. (**20 points.**) Not available in preview mode.
3. (**20 points.**) The free Green dyadic $\mathbf{\Gamma}_0(\mathbf{r},\mathbf{r}';\omega)$ satisfies the dyadic differential equation

$$\frac{c^2}{\omega^2}\left[\boldsymbol{\nabla}\boldsymbol{\nabla} - \mathbf{1}\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\right]\cdot\mathbf{\Gamma}_0(\mathbf{r},\mathbf{r}';\omega) = \mathbf{1}\delta^{(3)}(\mathbf{r}-\mathbf{r}'). \tag{1}$$

   (a) Show that the divergence of the free Green dyadic is

$$\boldsymbol{\nabla}\cdot\mathbf{\Gamma}_0(\mathbf{r},\mathbf{r}';\omega) = -\boldsymbol{\nabla}\delta^{(3)}(\mathbf{r}-\mathbf{r}'). \tag{2}$$

   (b) Substitute the divergence in the dyadic differential equation and derive

$$-\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\mathbf{\Gamma}_0(\mathbf{r},\mathbf{r}';\omega) = \left(\boldsymbol{\nabla}\boldsymbol{\nabla} + \frac{\omega^2}{c^2}\mathbf{1}\right)\delta^{(3)}(\mathbf{r}-\mathbf{r}'). \tag{3}$$

   (c) Construct the differential equation

$$-(\nabla^2 + k^2)G_0(\mathbf{r},\mathbf{r}';\omega) = \delta^{(3)}(\mathbf{r}-\mathbf{r}') \tag{4}$$

   for the Green function $G_0(\mathbf{r},\mathbf{r}';\omega)$, where

$$k = \frac{\omega}{c}. \tag{5}$$

   The free Green function has the (causal) solution

$$G_0(\mathbf{r}-\mathbf{r}';\omega) = \frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{6}$$

   Show that the free Green dyadic can be expressed in terms of the free Green function as

$$\mathbf{\Gamma}_0(\mathbf{r},\mathbf{r}';\omega) = \left[\boldsymbol{\nabla}\boldsymbol{\nabla} + k^2\mathbf{1}\right]G_0(\mathbf{r},\mathbf{r}';\omega) \tag{7}$$

(d) Evaluate the gradient operators and show that

$$\boldsymbol{\Gamma}_0(\mathbf{r};\omega) = \frac{e^{ikr}}{4\pi r^3}\Big[-u(ikr)\mathbf{1} + v(ikr)\hat{\mathbf{r}}\hat{\mathbf{r}}\Big], \tag{8}$$

where

$$u(x) = 1 - x + x^2, \tag{9a}$$
$$v(x) = 3 - 3x + x^2. \tag{9b}$$

4. (**20 points.**) The free Green dyadic $\boldsymbol{\Gamma}_0$ can be expressed in terms of the free Green function $G_0$ as

$$\boldsymbol{\Gamma}_0(\mathbf{r},\mathbf{r}';\omega) = \big[\boldsymbol{\nabla}\boldsymbol{\nabla} + k^2\mathbf{1}\big]G_0(\mathbf{r},\mathbf{r}';\omega), \tag{10}$$

where

$$G_0(\mathbf{r}-\mathbf{r}';\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{11}$$

In the far-field approximation,

$$r' \ll r, \tag{12}$$

when the observation point $\mathbf{r}$ is very far relative to the source point $\mathbf{r}'$, show that

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2 + r'^2 - 2rr'} \sim r - \hat{\mathbf{r}}\cdot\mathbf{r}'. \tag{13}$$

Thus, in the far-field asymptotic limit show that

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \to \frac{e^{ikr}}{4\pi r}\,e^{-i\mathbf{k}'\cdot\mathbf{r}'}, \tag{14}$$

where we introduced the notation

$$\mathbf{k}' = k\,\hat{\mathbf{r}}. \tag{15}$$

Further, the far-field approximation allows the replacement

$$\boldsymbol{\nabla} \to i\mathbf{k}'. \tag{16}$$

Thus, in the far-field approximation show that

$$(\boldsymbol{\nabla}\boldsymbol{\nabla} + k^2\mathbf{1}) \to (\mathbf{1} - \hat{\mathbf{r}}\hat{\mathbf{r}})k^2 = -\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{1})k^2, \tag{17}$$

which projects vectors in the plane normal to the radial direction. Thus, show that the free Green dyadic in the far-field approximation takes the form

$$\boldsymbol{\Gamma}_0(\mathbf{r},\mathbf{r}';\omega) = -\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{1})\frac{k^2}{4\pi}\frac{e^{ikr}}{r}\,e^{-i\mathbf{k}'\cdot\mathbf{r}'}. \tag{18}$$

5. (**20 points.**) The scattering amplitude is given by

$$f(\theta,\phi,\omega) = -\frac{k^2}{4\pi}\chi(\mathbf{k}' - \mathbf{k}, \omega), \tag{19}$$

where $\chi(\mathbf{q},\omega)$ is the Fourier transform of $\chi(\mathbf{r},\omega)$,

$$\chi(\mathbf{q},\omega) = \int d^3r\, e^{i\mathbf{q}\cdot\mathbf{r}}.\chi(\mathbf{r},\omega) \tag{20}$$

If the obstacles are confined on a plane, say $z = 0$, then it is convenient to define polarizability per unit area $\boldsymbol{\lambda} = \boldsymbol{\alpha}/\text{Area}$,

$$\boldsymbol{\chi}(\mathbf{r},\omega) = 4\pi\boldsymbol{\lambda}(\mathbf{s})\,\delta(z), \tag{21}$$

where the $\delta$-function has been used to describe the assumption that the obstacles in a thin film are confined to a plane, $z = 0$ here. Once the obstacles are restricted to be on a plane, we can choose the direction of incidence $\mathbf{k}$ of the plane wave to be normal to the plane. That is, $\mathbf{k}\cdot\mathbf{s} = 0$, where $\mathbf{s}$ are the positions of the point obstacles on the plane. Further, notice that in this special case the electric field $\mathbf{E}_0$ is independent of the position $\mathbf{s}$. Using these considerations show that the scattering amplitude, for isotropic polarizabilities, is given by

$$f(\theta,\phi,\omega) = -k^2\int d^2s\, e^{ik\hat{\mathbf{r}}\cdot\mathbf{s}}\lambda(\mathbf{s}). \tag{22}$$

For a disc of radius $R$ centered at position $\mathbf{s}_0$ with uniform polarizability per unit area $\lambda$ complete the integrals to obtain

$$f(\theta,\phi,\omega) = -\lambda k^2\pi R^2 2\frac{J_1(kR\sin\theta)}{kR\sin\theta}\,e^{ik\hat{\mathbf{r}}\cdot\mathbf{s}_0}. \tag{23}$$

Hint: Use the integral representation of zeroth order Bessel function of the first kind

$$J_0(t) = \int_0^{2\pi}\frac{d\phi}{2\pi}\,e^{it\cos\phi} \tag{24}$$

and the identity

$$\int_0^b tdtJ_0(t) = bJ_1(b). \tag{25}$$

Note the limiting value

$$\lim_{x\to 0}\frac{J_1(x)}{x} = \frac{1}{2}, \tag{26}$$

which guarantees a well defined value for the scattering amplitude at $\theta = 0$. We observe the interesting feature that the scattering amplitude at $\theta = 0$ is entirely given by the area of the disc.