Midterm Exam No. 02 (2022 Spring)

PHYS 520B: ELECTROMAGNETIC THEORY

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1. (**20 points.**) Prove the following. If u^α is a time-like vector and $u^\alpha a_\alpha = 0$, then a^α is necessarily space-like.

2. (**20 points.**) Lorentz transformation relates the energy E and momentum $\mathbf{p}$ of a particle when measured in different frames. For example, for the special case when the relative velocity and the velocity of the particle are parallel we have

$$\begin{pmatrix} E'/c \\ p' \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E/c \\ p \end{pmatrix}. \tag{1}$$

   Photons are massless spin 1 particles whose energy and momentum are $E = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k}$, such that $\omega = kc$. Thus, derive the relativistic Doppler effect formula

$$\omega' = \omega\sqrt{\frac{1+\beta}{1-\beta}}. \tag{2}$$

3. (**20 points.**) If the motion of a non-relativistic particle is such that it does not change the kinetic energy of the particle, we have

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = 0. \tag{3}$$

   Show that this imples

$$\mathbf{v}\cdot\mathbf{a} = 0. \tag{4}$$

   This is achieved when the acceleration $a = 0$ or in the case of uniform circular motion. Starting from Eq. (4) show that the relativistic generalization of kinetic energy $E = mc^2\gamma$ is also conserved, that is,

$$\frac{d}{dt}(mc^2\gamma) = 0. \tag{5}$$

   Observe that

$$\boldsymbol{\beta}\cdot\mathbf{a} = \frac{d}{dt}\left(\frac{\beta^2}{2}\right) = -\frac{1}{2}\frac{d}{dt}\frac{1}{\gamma^2} = \frac{1}{\gamma^3}\frac{d\gamma}{dt}. \tag{6}$$

4. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration α is described by equation of a hyperbola

$$z^2 - c^2t^2 = z_0^2, \qquad z_0 = \frac{c^2}{\alpha}. \tag{7}$$

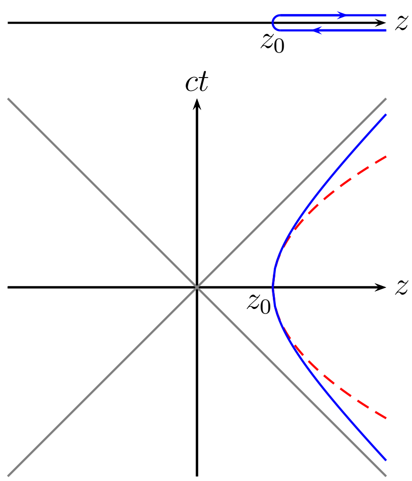

Figure 1: Problem 4

(a) This represents the world-line of a particle thrown from $z > z_0$ at $t < 0$ towards $z = z_0$ in region of constant (proper) acceleration α as described by the bold (blue) curve in the space-time diagram in Figure 4. In contrast a Newtonian particle moving with constant acceleration α is described by equation of a parabola

$$z - z_0 = \frac{1}{2}\alpha t^2 \tag{8}$$

as described by the dashed (red) curve in the space-time diagram in Figure 4. Show that the hyperbolic curve

$$z = z_0\sqrt{1 + \frac{c^2t^2}{z_0^2}} \tag{9}$$

in regions that satisfy

$$t \ll \frac{c}{\alpha} \tag{10}$$

is approximately the parabolic curve

$$z = z_0 + \frac{1}{2}\alpha t^2 + \ldots. \tag{11}$$

(b) Recognize that the proper acceleration α does not have an upper bound.

(c) A large acceleration is achieved by taking above turn while moving very fast. Thus, turning around while moving close to the speed of light c should achieve the highest acceleration. Show that $\alpha \to \infty$ corresponding to $z_0 \to 0$ represents this scenario. What is the equation of motion of a particle moving with infinite proper acceleration. Qualitatively, to gain insight, plot world-lines of particles moving with $\alpha = c^2/z_0$, $\alpha = 10c^2/z_0$, and $\alpha = 100c^2/z_0$.