Homework No. 08 (2022 Spring)

PHYS 520B: ELECTROMAGNETIC THEORY

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Due date: Thursday, 2022 Apr 7, 12.30pm

1. (**20 points.**) A charged particle with charge q moves on the z-axis with constant speed v, $\beta = v/c$. The electric and magnetic field generated by this charged particle is given by

$$\mathbf{E}(\mathbf{r},t) = (1-\beta^2)\frac{q}{4\pi\varepsilon_0}\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z-vt)\hat{\mathbf{k}}}{\left[(x^2+y^2)(1-\beta^2)+(z-vt)^2\right]^{\frac{3}{2}}}, \tag{1a}$$

$$c\mathbf{B}(\mathbf{r},t) = \beta(1-\beta^2)\frac{q}{4\pi\varepsilon_0}\frac{-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}}{\left[(x^2+y^2)(1-\beta^2)+(z-vt)^2\right]^{\frac{3}{2}}}. \tag{1b}$$

Evaluate the electromagnetic momentum density for this configuration by evaluating

$$\mathbf{G}(\mathbf{r},t) = \varepsilon_0 \mathbf{E}(\mathbf{r},t) \times \mathbf{B}(\mathbf{r},t). \tag{2}$$

2. (**20 points.**) The electric and magnetic field generated by a particle with charge q moving along the z axis with speed v, $\beta = v/c$, can be expressed in the form

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{\left[x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z-vt)\hat{\mathbf{k}}\right]}{(x^2+y^2)}\frac{(x^2+y^2)(1-\beta^2)}{\left[(x^2+y^2)(1-\beta^2)+(z-vt)^2\right]^{\frac{3}{2}}}, \tag{3a}$$

$$c\mathbf{B}(\mathbf{r},t) = \boldsymbol{\beta} \times \mathbf{E}(\mathbf{r},t). \tag{3b}$$

(a) Consider the distribution

$$\delta(x) = \lim_{\epsilon\to 0}\frac{1}{2}\frac{\epsilon}{(x^2+\epsilon)^{\frac{3}{2}}}. \tag{4}$$

Show that

$$\delta(x) \begin{cases} \to \frac{1}{2\sqrt{\epsilon}} \to \infty, & \text{if} \quad x = 0, \\ \to \frac{\epsilon}{2x^3} \to 0, & \text{if} \quad x \neq 0. \end{cases} \tag{5}$$

Further, show that

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1. \tag{6}$$

(b) Thus, verify that the electric and magnetic field of a charge approaching the speed of light can be expressed in the form

$$\mathbf{E}(\mathbf{r},t) = \frac{2q}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\rho}}}{\rho}\,\delta(z-ct), \tag{7a}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\frac{2q}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\phi}}}{\rho}\,\delta(z-ct) = 2q\left(\frac{\mu_0 c}{4\pi}\right)\frac{\hat{\boldsymbol{\phi}}}{\rho}\,\delta(z-ct), \tag{7b}$$

where $\boldsymbol{\rho} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ and $\rho = \sqrt{x^2 + y^2}$, $\boldsymbol{\phi} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$, and $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ are the associated unit vectors. These fields are confined on the $z = ct$ plane moving with speed c. Illustrate this configuration of fields using a diagram.

(c) To confirm that the above confined fields are indeed solutions to the Maxwell equations, verify the following:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{\varepsilon_0} q \delta^{(2)}(\boldsymbol{\rho}) \delta(z - ct), \tag{8a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{8b}$$

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{8c}$$

$$\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 q c \hat{\mathbf{z}} \delta^{(2)}(\boldsymbol{\rho}) \delta(z - ct). \tag{8d}$$

This is facilitated by writing

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}_\rho + \hat{\mathbf{z}} \frac{\partial}{\partial z}, \tag{9}$$

and accomplished by using the following identities:

$$\boldsymbol{\nabla}_\rho \cdot \left(\frac{\hat{\boldsymbol{\rho}}}{\rho} \right) = 2\pi \delta^{(2)}(\boldsymbol{\rho}), \qquad \boldsymbol{\nabla}_\rho \times \left(\frac{\hat{\boldsymbol{\rho}}}{\rho} \right) = 0, \tag{10a}$$

$$\boldsymbol{\nabla}_\rho \cdot \left(\frac{\hat{\boldsymbol{\phi}}}{\rho} \right) = 0, \qquad \boldsymbol{\nabla}_\rho \times \left(\frac{\hat{\boldsymbol{\phi}}}{\rho} \right) = \hat{\mathbf{z}}\, 2\pi \delta^{(2)}(\boldsymbol{\rho}). \tag{10b}$$