Midterm Exam No. 02 (2024 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Date: 2024 Apr 4

1. (**20 points.**) Given
$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{1}$$
and
$$\boldsymbol{\phi} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}, \tag{2}$$
determine a, in terms of x, y, and z, such that
$$\hat{\mathbf{z}} \times \hat{\mathbf{r}} = a\,\hat{\boldsymbol{\phi}} \tag{3}$$
is an identity.

2. (**20 points.**) Consider the Hamiltonian
$$H = H(\mathbf{r}, \mathbf{p}, t), \tag{4}$$
which satisfies the Hamilton equations of motion
$$\frac{d\mathbf{r}}{dt} = \frac{\partial H}{\partial \mathbf{p}}, \tag{5a}$$
$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{r}}. \tag{5b}$$
The p-Lagrangian is constructed usng the definition
$$L_p = -\mathbf{r} \cdot \frac{d\mathbf{p}}{dt} - H(\mathbf{r}, \mathbf{p}, t). \tag{6}$$
Investigate the dependence of the p-Lagrangian on the variable $\mathbf{r}$ by evaluating the partial derivative with respect to it. That is, evaluate
$$\frac{\partial L_p}{\partial \mathbf{r}}. \tag{7}$$

3. (**20 points.**) A mass m slides down a frictionless ramp that is inclined at an angle θ with respect to the horizontal. Assume uniform acceleration due to gravity g in the vertical downward direction. In terms of a suitable dynamical variable write a Lagrangian that describes the motion of the mass.

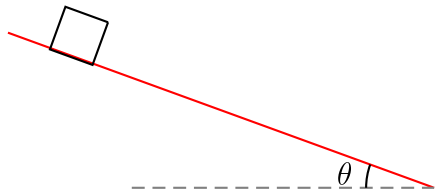

Figure 1: Problem 3.

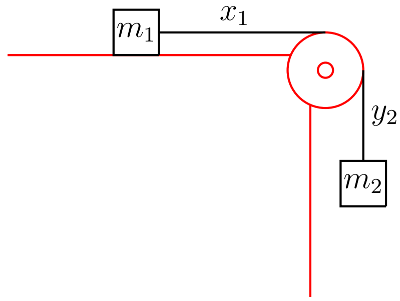

Figure 2: Problem 4

4. (**20 points.**) A mass m_2 is connected to another mass m_1 by a massless (inextensible) string passing over a massless pulley, as described in Figure 2. Surfaces are frictionless. Massless pulley implies that tension in the string on both sides of the pulley is the same, say T. Further, the string being inextensible implies that the magnitude of the accelerations of both the masses are the same.

   (a) Let lengths x_1 and y_2 be positive distances from the pulley to the masses such that the accelerations $a_1 = \ddot{x}_1$ and $a_2 = \ddot{y}_2$ satisfy $a_2 = -a_1 = a$. Using Newton's law determine the equations of motion for the masses to be

$$m_2 g - T = m_2 a, \tag{8a}$$
$$T = m_1 a. \tag{8b}$$

   Thus, show that

$$\text{Equation of motion:} \qquad a = \left(\frac{m_2}{m_1 + m_2}\right) g, \tag{9a}$$
$$\text{Equation of constraint:} \qquad T = \left(\frac{m_2}{m_1 + m_2}\right) m_1 g. \tag{9b}$$

   (b) The constraint among the dynamical variables x_1 and y_2 is

$$x_1 + y_2 = L, \tag{10}$$

where L is the total length of the string connecting the two masses. Show that the Lagrangian describing the motion can be expressed in terms of a single dynamical variable, say y_2, as

$$L(y_2, \dot{y}_2) = \frac{1}{2}(m_1 + m_2)\dot{y}_2^2 + m_2 g y_2. \tag{11}$$

Find the corresponding Euler-Lagrange equation.

(c) Using the idea of Lagrange multiplier construct a Lagrangian $L(x_1, y_2, \dot{x}_1, \dot{y}_2)$ which reproduces both the equation of motion and the equation of constraint.

5. (**20 points.**) In the small angle approximation the equations of motion for a double pendulum, see Figure 5, reduce to

$$\ddot{\theta}_1 + \omega_1^2\theta_1 + \frac{\alpha}{\beta}\ddot{\theta}_2 = 0, \tag{12a}$$

$$\ddot{\theta}_2 + \omega_2^2\theta_2 + \beta\ddot{\theta}_1 = 0, \tag{12b}$$

where

$$\omega_1^2 = \frac{g}{a_1}, \quad \omega_2^2 = \frac{g}{a_2}, \quad \alpha = \frac{m_2}{m_1 + m_2}, \quad \beta = \frac{a_1}{a_2} = \frac{\omega_2^2}{\omega_1^2}. \tag{13}$$

Note that $0 \leq \alpha \leq 1$. Determine the normal modes for the above motion. What happens to the normal modes when we double β?

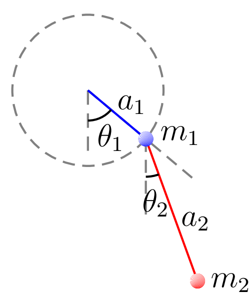

Figure 3: Problem 5.