Homework No. 08 (2024 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2024 Apr 2, 4.30pm

1. (**20 points.**) The Hamiltonian is defined by the relation

$$H(p_i, q_i, t) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t). \tag{1}$$

Show that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{2}$$

2. (**20 points.**) Consider infinitesimal rigid translation in space, described by

$$\delta \mathbf{r} = \delta \boldsymbol{\epsilon}, \quad \delta \mathbf{p} = 0, \quad \delta t = 0, \tag{3}$$

where $\delta \boldsymbol{\epsilon}$ is independent of position and time.

(a) Show that the change in the action due to the above translation is

$$\frac{\delta W}{\delta \boldsymbol{\epsilon}} = -\int_{t_1}^{t_2} dt \frac{\partial H}{\partial \mathbf{r}}. \tag{4}$$

(b) Show, separately, that the change in the action under the above translation is also given by

$$\frac{\delta W}{\delta \boldsymbol{\epsilon}} = \int_{t_1}^{t_2} dt \frac{d\mathbf{p}}{dt} = \mathbf{p}(t_2) - \mathbf{p}(t_1). \tag{5}$$

(c) The system is defined to have translational symmetry when the action does not change under rigid translation. Show that a system has translation symmetry when

$$-\frac{\partial H}{\partial \mathbf{r}} = 0. \tag{6}$$

That is, when the Hamiltonian is independent of position. Or, when the force $\mathbf{F} = -\partial H/\partial \mathbf{r} = 0$.

(d) Deduce that the linear momentum is conserved, that is,

$$\mathbf{p}(t_1) = \mathbf{p}(t_2), \tag{7}$$

when the action has translation symmetry.

3. (**20 points.**) Consider infinitesimal rigid translation in time, described by

$$\delta\mathbf{r} = 0, \quad \delta\mathbf{p} = 0, \quad \delta t = \delta\epsilon, \tag{8}$$

where $\delta\epsilon$ is independent of position and time.

(a) Show that the change in the action due to the above translation is

$$\frac{\delta W}{\delta\epsilon} = -\int_{t_1}^{t_2} dt \frac{\partial H}{\partial t}. \tag{9}$$

(b) Show, separately, that the change in the action under the above translation is also given by

$$\frac{\delta W}{\delta\epsilon} = -\int_{t_1}^{t_2} dt \frac{dH}{dt} = -H(t_2) + H(t_1). \tag{10}$$

(c) The system is defined to have translational symmetry when the action does not change under rigid translation. Show that a system has translation symmetry when

$$-\frac{\partial H}{\partial t} = 0. \tag{11}$$

That is, when the Hamiltonian is independent of time.

(d) Deduce that the Hamiltonian is conserved, that is,

$$H(t_1) = H(t_2), \tag{12}$$

when the action has translation symmetry.

4. (**20 points.**) A general rotation in 3-dimensions can be written in terms of consecutive rotations about x, y, and z axes,

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{13}$$

For infinitesimal rotations we use

$$\cos\theta_i \sim 1, \tag{14a}$$

$$\sin\theta_i \sim \theta_i \to \delta\theta_i, \tag{14b}$$

to obtain

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & \delta\theta_3 & -\delta\theta_2 \\ -\delta\theta_3 & 1 & \delta\theta_1 \\ \delta\theta_2 & -\delta\theta_1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{15}$$

Show that this corresponds to the vector relation

$$\mathbf{r}' = \mathbf{r} - \delta\boldsymbol{\theta} \times \mathbf{r}, \tag{16}$$

where

$$\mathbf{r} = x_1\hat{\mathbf{x}} + x_2\hat{\mathbf{y}} + x_3\hat{\mathbf{z}}, \tag{17a}$$

$$\delta\boldsymbol{\theta} = \delta\theta_1\hat{\mathbf{x}} + \delta\theta_2\hat{\mathbf{y}} + \delta\theta_3\hat{\mathbf{z}}. \tag{17b}$$

As a particular example, verify that a rotation about the direction $\hat{\mathbf{z}}$ by an infinitesimal (azimuth) angle $\delta\phi$ is described by

$$\delta\boldsymbol{\theta} = \hat{\mathbf{z}}\,\delta\phi. \tag{18}$$

The corresponding infinitesimal transformation in $\mathbf{r}$ is given by

$$\delta\mathbf{r} = \delta\phi\,\hat{\mathbf{z}} \times \mathbf{r} = \hat{\boldsymbol{\phi}}\,\rho\delta\phi, \tag{19}$$

where ρ and ϕ are the cylindrical coordinates defined as

$$\hat{\mathbf{z}} \times \mathbf{r} = \boldsymbol{\phi} \quad \text{and} \quad |\hat{\mathbf{z}} \times \mathbf{r}| = \rho. \tag{20}$$

Observe that, in rectangular coordinates $\rho\hat{\boldsymbol{\phi}} = x\hat{\mathbf{y}} - y\hat{\mathbf{x}}$.

5. (**20 points.**) Consider infinitesimal rigid rotation, described by

$$\delta\mathbf{r} = \delta\boldsymbol{\omega} \times \mathbf{r}, \quad \delta\mathbf{p} = \delta\boldsymbol{\omega} \times \mathbf{p}, \quad \delta t = 0, \tag{21}$$

where $d\delta\boldsymbol{\omega}/dt = 0$.

(a) Show that the variation in the action under the above rotation is

$$\frac{\delta W}{\delta\boldsymbol{\omega}} = \int_{t_1}^{t_2} dt \left[\mathbf{r} \times \frac{\partial L}{\partial \mathbf{r}} + \mathbf{p} \times \frac{\partial L}{\partial \mathbf{p}}\right] \tag{22}$$

or

$$\frac{\delta W}{\delta\boldsymbol{\omega}} = -\int_{t_1}^{t_2} dt \left[\mathbf{r} \times \frac{\partial H}{\partial \mathbf{r}} + \mathbf{p} \times \frac{\partial H}{\partial \mathbf{p}}\right]. \tag{23}$$

(b) Show, separately, that the change in the action under the above rotation is also given by

$$\frac{\delta W}{\delta\boldsymbol{\omega}} = \int_{t_1}^{t_2} dt \frac{d\mathbf{L}}{dt} = \mathbf{L}(t_2) - \mathbf{L}(t_1), \tag{24}$$

where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the angular momentum.

(c) The system is defined to have rotational symmetry when the action does not change under rigid rotation. Show that a system has rotation symmetry when

$$\mathbf{r} \times \frac{\partial L}{\partial \mathbf{r}} = 0 \quad \text{and} \quad \mathbf{p} \times \frac{\partial L}{\partial \mathbf{p}} = 0, \tag{25}$$

or

$$\mathbf{r} \times \frac{\partial H}{\partial \mathbf{r}} = 0 \quad \text{and} \quad \mathbf{p} \times \frac{\partial H}{\partial \mathbf{p}} = 0. \tag{26}$$

Show that this corresponds to

$$\frac{\partial L}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \phi} = 0, \tag{27}$$

or

$$\frac{\partial H}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial H}{\partial \phi} = 0. \tag{28}$$

That is, when the Lagrangian is independent of angular coordinates θ and ϕ.

(d) Deduce that the anglular momentum is conserved, that is,

$$\mathbf{L}(t_1) = \mathbf{L}(t_2), \tag{29}$$

when the action has rotational symmetry.