Homework No. 09 (2024 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2024 Apr 17, 4.30pm

1. (**20 points.**) For two functions

$$A = A(\mathbf{x}, \mathbf{p}, t), \tag{1a}$$
$$B = B(\mathbf{x}, \mathbf{p}, t), \tag{1b}$$

the Poisson braket with respect to the canonical variables $\mathbf{x}$ and $\mathbf{p}$ is defined as

$$\big[A, B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} \equiv \frac{\partial A}{\partial \mathbf{x}} \cdot \frac{\partial B}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}} \cdot \frac{\partial B}{\partial \mathbf{x}}. \tag{2}$$

Show that the Poisson braket satisfies the conditions for a Lie algebra. That is, show that

(a) Antisymmetry:

$$\big[A, B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = -\big[B, A\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}. \tag{3}$$

(b) Bilinearity: (a and b are numbers.)

$$\big[aA + bB, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = a\big[A, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + b\big[B, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}. \tag{4}$$

Further show that

$$\big[AB, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = A\big[B, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \big[A, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} B. \tag{5}$$

(c) Jacobi's identity:

$$\Big[A, \big[B, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \Big[B, \big[C, A\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \Big[C, \big[A, B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0. \tag{6}$$

2. (**20 points.**) Show that the commutator of two matrices,

$$\big[\mathbf{A}, \mathbf{B}\big] \equiv \mathbf{AB} - \mathbf{BA}, \tag{7}$$

satisfies the conditions for a Lie algebra, as does the Poisson bracket. In particular show that

(a) Antisymmetry:

$$\big[\mathbf{A}, \mathbf{B}\big] = -\big[\mathbf{B}, \mathbf{A}\big]. \tag{8}$$

(b) Bilinearity: (a and b are numbers.)

$$\big[a\mathbf{A} + b\mathbf{B}, \mathbf{C}\big] = a\big[\mathbf{A}, \mathbf{C}\big] + b\big[\mathbf{B}, \mathbf{C}\big]. \tag{9}$$

Further show that

$$\big[\mathbf{A}\mathbf{B}, \mathbf{C}\big] = \mathbf{A}\big[\mathbf{B}, \mathbf{C}\big] + \big[\mathbf{A}, \mathbf{C}\big]\mathbf{B}. \tag{10}$$

(c) Jacobi's identity:

$$\big[\mathbf{A}, \big[\mathbf{B}, \mathbf{C}\big]\big] + \big[\mathbf{B}, \big[\mathbf{C}, \mathbf{A}\big]\big] + \big[\mathbf{C}, \big[\mathbf{A}, \mathbf{B}\big]\big] = 0. \tag{11}$$

3. (**20 points.**) Show that the vector product of two vectors, in this problem denoted using

$$\big[\mathbf{A}, \mathbf{B}\big]_v \equiv \mathbf{A} \times \mathbf{B}, \tag{12}$$

satisfies the conditions for a Lie algebra, as does the Poisson bracket. In particular show that

(a) Antisymmetry:

$$\big[\mathbf{A}, \mathbf{B}\big]_v = -\big[\mathbf{B}, \mathbf{A}\big]_v. \tag{13}$$

(b) Bilinearity: (a and b are numbers.)

$$\big[a\mathbf{A} + b\mathbf{B}, \mathbf{C}\big]_v = a\big[\mathbf{A}, \mathbf{C}\big]_v + b\big[\mathbf{B}, \mathbf{C}\big]_v. \tag{14}$$

Further show that

$$\big[\mathbf{A} \times \mathbf{B}, \mathbf{C}\big]_v = \mathbf{A} \times \big[\mathbf{B}, \mathbf{C}\big]_v + \big[\mathbf{A}, \mathbf{C}\big]_v \times \mathbf{B}. \tag{15}$$

(c) Jacobi's identity:

$$\big[\mathbf{A}, \big[\mathbf{B}, \mathbf{C}\big]_v\big]_v + \big[\mathbf{B}, \big[\mathbf{C}, \mathbf{A}\big]_v\big]_v + \big[\mathbf{C}, \big[\mathbf{A}, \mathbf{B}\big]_v\big]_v = 0. \tag{16}$$

4. (**20 points.**) Given F and G are constants of motion, that is

$$\big[F, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0 \qquad \text{and} \qquad \big[G, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0. \tag{17}$$

Then, using Jacobi's identity, show that $\big[F, G\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}$ is also a constant of motion. Thus, conclude the following:

(a) If L_x and L_y are constants of motion, then L_z is also a constant of motion.

(b) If p_x and L_z are constants of motion, then p_y is also a constant of motion.

5. (**20 points.**) (Refer Sec. 21 Dirac's QM book.)
The product rule for Poisson braket can be stated in the following different forms:

$$\big[A_1 A_2, B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = A_1\big[A_2, B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \big[A_1, B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} A_2, \tag{18a}$$

$$\big[A, B_1 B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = B_1\big[A, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \big[A, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} B_2. \tag{18b}$$

(a) Thus, evaluate, in two different ways,

$$\begin{aligned}\big[A_1A_2, B_1B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} &= A_1B_1\big[A_2, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + A_1\big[A_2, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}B_2 \\ &\quad + B_1\big[A_1, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}A_2 + \big[A_1, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}B_2A_2, \qquad (19a)\\ \big[A_1A_2, B_1B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} &= B_1A_1\big[A_2, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + B_1\big[A_1, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}A_2 \\ &\quad + A_1\big[A_2, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}B_2 + \big[A_1, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}A_2B_2. \qquad (19b)\end{aligned}$$

(b) Subtracting these results, obtain

$$(A_1B_1 - B_1A_1)\big[A_2, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \big[A_1, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}(A_2B_2 - B_2A_2). \qquad (20)$$

Thus, using the definition of the commutation relation,

$$\big[A, B\big] \equiv AB - BA, \qquad (21)$$

obtain the relation

$$\big[A_1, B_1\big]\big[A_2, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \big[A_1, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\big[A_2, B_2\big]. \qquad (22)$$

(c) Since this condition holds for A_1 and B_1 independent of A_2 and B_2, conclude that

$$\big[A_1, B_1\big] = i\hbar\big[A_1, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}, \qquad (23a)$$

$$\big[A_2, B_2\big] = i\hbar\big[A_2, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}, \qquad (23b)$$

where $i\hbar$ is necessarily a constant, independent of A_1, A_2, B_1, and B_2. This is the connection between the commutator braket in quantum mechanics and the Poisson braket in classical mechanics. If A's and B's are numbers, then, because their commutation relation is equal to zero, we necessairily have $\hbar = 0$. But, if the commutation relation of A's and B's is not zero, then finite values of $\hbar$ is allowed.

(d) Here the imaginary number $i = \sqrt{-1}$. Show that the constant $\hbar$ is a real number if we presume the Poisson braket to be real, and require the construction

$$C = \frac{1}{i}(AB - BA) \qquad (24)$$

to be Hermitian. Experiment dictates that $\hbar = h/2\pi$, where

$$h \sim 6.63 \times 10^{-34}\,\text{J}\cdot\text{s} \qquad (25)$$

is the Planck's constant with dimensions of action.