Homework No. 10 (2024 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2024 Apr 23, 4.30pm

1. (**20 points.**) (Refer Schwinger's QM, chapter 9) The Hamiltonian for a Kepler problem is

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{\alpha}{|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{1}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the positions of the two constituent particles of masses m_1 and m_2.

(a) Introduce the coordinates representing the center of mass, relative position, total momentum, and relative momentum:

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \quad \mathbf{r} = \mathbf{r_1} - \mathbf{r_2}, \quad \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \quad \mathbf{p} = \frac{m_2\mathbf{p}_1 - m_1\mathbf{p}_2}{m_1 + m_2}, \tag{2}$$

respectively, to rewrite the Hamiltonian as

$$H = \frac{P^2}{2M} + \frac{p^2}{2\mu} - \frac{\alpha}{r}, \tag{3}$$

where

$$M = m_1 + m_2, \qquad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{4}$$

(b) Show that Hamilton's equations of motion are given by

$$\frac{d\mathbf{R}}{dt} = \frac{\mathbf{P}}{M}, \quad \frac{d\mathbf{P}}{dt} = 0, \quad \frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{\mu}, \quad \frac{d\mathbf{p}}{dt} = -\frac{\alpha\mathbf{r}}{r^3}. \tag{5}$$

(c) Verify that the Hamiltonian H, the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and the Laplace-Runge-Lenz vector

$$\mathbf{A} = \frac{\mathbf{r}}{r} - \frac{\mathbf{p} \times \mathbf{L}}{\mu\alpha}, \tag{6}$$

are the three constants of motion for the Kepler problem. That is, show that

$$\frac{dH}{dt} = 0, \quad \frac{d\mathbf{L}}{dt} = 0, \quad \frac{d\mathbf{A}}{dt} = 0. \tag{7}$$

2. (**20 points.**) In the Kepler problem the orbit of a planet is a conic section

$$r(\phi) = \frac{r_0}{1 + e\cos(\phi - \phi_0)} \tag{8}$$

expressed in terms of the eccentricity e and distance r_0. Determine the constant ϕ_0 to be 0 by requiring the initial condition

$$r(0) = \frac{r_0}{1+e}. \tag{9}$$

This leads to

$$r(\pi) = \frac{r_0}{1-e}. \tag{10}$$

The distance r_0 is characterized by the fact that the effective potential

$$U_{\text{eff}}(r) = \frac{L_z^2}{2\mu r^2} - \frac{\alpha}{r} \tag{11}$$

is minimum at r_0. We used the definitions

$$r_0 = \frac{L_z^2}{\mu\alpha}, \qquad U_{\text{eff}}(r_0) = -\frac{\alpha}{2r_0}, \qquad e = \sqrt{1 - \frac{E}{U_{\text{eff}}(r_0)}}. \tag{12}$$

Thus, the orbit of a planet is completely determined by the energy E and the angular momentum L_z, which are constants of motion. The statement of conservation of angular momentum can be expressed in the form

$$dt = \frac{\mu}{L_z} r^2 d\phi, \tag{13}$$

which is convenient for evaluating the time elapsed in the motion. For the case of elliptic orbit, $U_{\text{eff}}(r_0) < E < 0$, show that the time period is given by

$$T = \frac{\mu}{L_z} \int_0^{2\pi} d\phi \frac{r_0^2}{(1+e\cos\phi)^2} = \frac{\mu r_0^2}{L_z} \frac{2\pi}{(1-e^2)^{\frac{3}{2}}}. \tag{14}$$

Show that at point '2' in Figure 2

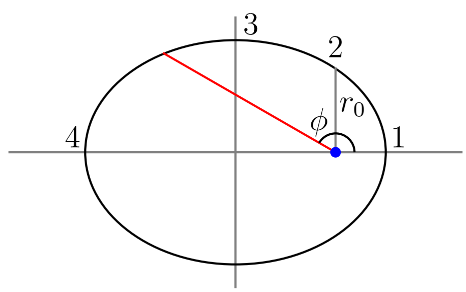

Figure 1: Elliptic orbit

$$\phi = \frac{\pi}{2}, \quad \text{and} \quad r = r_0. \tag{15}$$

The time taken to go from '1' to '2' is given by (need not be proved here)

$$t_{1\to 2} = \frac{\mu}{L_z}\int_0^{\frac{\pi}{2}} d\phi \frac{r_0^2}{(1+e\cos\phi)^2} = \frac{T}{4}\left(\frac{4}{\pi}\tan^{-1}\sqrt{\frac{1-e}{1+e}} - \frac{2e}{\pi}\sqrt{1-e^2}\right). \tag{16}$$

Evaluate $t_{1\to 2}$ for $e=0$ and $e=1$. Show that at point '3' in Figure 2

$$\phi = \pi - \tan^{-1}\left(\frac{\sqrt{1-e^2}}{e}\right), \quad \text{and} \quad r = a. \tag{17}$$

The time taken to go from '1' to '3' is given by (need not be proved here)

$$t_{1\to 3} = \frac{\mu}{L_z}\int_0^{\pi-\tan^{-1}\left(\frac{\sqrt{1-e^2}}{e}\right)} d\phi \frac{r_0^2}{(1+e\cos\phi)^2} = \frac{T}{4}\left(1 - \frac{2e}{\pi}\right). \tag{18}$$

Similarly, the time taken to go from '3' to '4' is given by (need not be proved here)

$$t_{3\to 4} = \frac{\mu}{L_z}\int_{\pi-\tan^{-1}\left(\frac{\sqrt{1-e^2}}{e}\right)}^{\pi} d\phi \frac{r_0^2}{(1+e\cos\phi)^2} = \frac{T}{4}\left(1 + \frac{2e}{\pi}\right). \tag{19}$$

Evaluate the time elapsed in the above cases for $e\to 0$ and $e\to 1$. The eccentricity e of Earth's orbit is 0.0167 and timeperiod T is 365 days. Thus, calculate

$$t_{1\to 3} - t_{1\to 2} \tag{20}$$

for Earth in units of days.