

Homework No. 01 (Fall 2024)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Monday, 2024 Aug 26, 4.30pm

1. **(20 points.)** Verify the following relations:

$$\delta_{ij} = \delta_{ji}, \quad (1a)$$

$$\delta_{ii} = 3, \quad (1b)$$

$$\delta_{ik}\delta_{kj} = \delta_{ij}, \quad (1c)$$

$$\delta_{im}B_m = B_i, \quad (1d)$$

$$\varepsilon_{ijk} = -\varepsilon_{ikj} = \varepsilon_{kij}, \quad (1e)$$

$$\varepsilon_{iik} = 0, \quad (1f)$$

$$\delta_{ij}\varepsilon_{ijk} = 0. \quad (1g)$$

2. **(20 points.)** In three dimensions the Levi-Civita symbol is given in terms of the determinant of the Kronecker δ -functions,

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \quad (2a)$$

$$= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}). \quad (2b)$$

Using the above identity show that

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}, \quad (3a)$$

$$\varepsilon_{ijk}\varepsilon_{ijn} = 2\delta_{kn}, \quad (3b)$$

$$\varepsilon_{ijk}\varepsilon_{ijk} = 6. \quad (3c)$$

3. **(20 points.)** Use index notation or dyadic notation to show that

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}, \quad (4a)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B}), \quad (4b)$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A}(\nabla \cdot \mathbf{B}) - (\nabla \cdot \mathbf{A})\mathbf{B} - (\mathbf{A} \cdot \nabla)\mathbf{B}. \quad (4c)$$

4. **(20 points.)** Consider the dyadic construction

$$\mathbf{T} = \mathbf{E} \mathbf{B} \quad (5)$$

built using the vector fields,

$$\mathbf{E} = E_x \hat{\mathbf{x}} + E_y \hat{\mathbf{y}}, \quad (6a)$$

$$\mathbf{B} = B \hat{\mathbf{y}}. \quad (6b)$$

Evaluate the following components of the dyadic:

$$\hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} = \quad \hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{y}} = \quad \hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{z}} = \quad (7a)$$

$$\hat{\mathbf{y}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} = \quad \hat{\mathbf{y}} \cdot \mathbf{T} \cdot \hat{\mathbf{y}} = \quad \hat{\mathbf{y}} \cdot \mathbf{T} \cdot \hat{\mathbf{z}} = \quad (7b)$$

$$\hat{\mathbf{z}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} = \quad \hat{\mathbf{z}} \cdot \mathbf{T} \cdot \hat{\mathbf{y}} = \quad \hat{\mathbf{z}} \cdot \mathbf{T} \cdot \hat{\mathbf{z}} = \quad (7c)$$

Evaluate the scalars

$$\text{Tr}(\mathbf{T}) = T_{ii}, \quad (8a)$$

$$\text{Tr}(\mathbf{T} \cdot \mathbf{T}) = T_{ij}T_{ji}, \quad (8b)$$

$$\text{Tr}(\mathbf{T} \cdot \mathbf{T} \cdot \mathbf{T}) = T_{ij}T_{jk}T_{ki}. \quad (8c)$$

Evaluate the following vector field constructions:

$$\hat{\mathbf{x}} \cdot \mathbf{T} = \quad \hat{\mathbf{y}} \cdot \mathbf{T} = \quad \hat{\mathbf{z}} \cdot \mathbf{T} = \quad (9a)$$

$$\mathbf{T} \cdot \hat{\mathbf{x}} = \quad \mathbf{T} \cdot \hat{\mathbf{y}} = \quad \mathbf{T} \cdot \hat{\mathbf{z}} = \quad (9b)$$

$$\hat{\mathbf{x}} \times \mathbf{T} \cdot \hat{\mathbf{x}} = \quad \hat{\mathbf{y}} \times \mathbf{T} \cdot \hat{\mathbf{x}} = \quad \hat{\mathbf{z}} \times \mathbf{T} \cdot \hat{\mathbf{x}} = \quad (9c)$$

$$\hat{\mathbf{x}} \times \mathbf{T} \cdot \hat{\mathbf{y}} = \quad \hat{\mathbf{y}} \times \mathbf{T} \cdot \hat{\mathbf{y}} = \quad \hat{\mathbf{z}} \times \mathbf{T} \cdot \hat{\mathbf{y}} = \quad (9d)$$

$$\hat{\mathbf{x}} \times \mathbf{T} \cdot \hat{\mathbf{z}} = \quad \hat{\mathbf{y}} \times \mathbf{T} \cdot \hat{\mathbf{z}} = \quad \hat{\mathbf{z}} \times \mathbf{T} \cdot \hat{\mathbf{z}} = \quad (9e)$$

$$\hat{\mathbf{x}} \cdot \mathbf{T} \times \hat{\mathbf{x}} = \quad \hat{\mathbf{x}} \cdot \mathbf{T} \times \hat{\mathbf{y}} = \quad \hat{\mathbf{x}} \cdot \mathbf{T} \times \hat{\mathbf{z}} = \quad (9f)$$

$$\hat{\mathbf{y}} \cdot \mathbf{T} \times \hat{\mathbf{x}} = \quad \hat{\mathbf{y}} \cdot \mathbf{T} \times \hat{\mathbf{y}} = \quad \hat{\mathbf{y}} \cdot \mathbf{T} \times \hat{\mathbf{z}} = \quad (9g)$$

$$\hat{\mathbf{z}} \cdot \mathbf{T} \times \hat{\mathbf{x}} = \quad \hat{\mathbf{z}} \cdot \mathbf{T} \times \hat{\mathbf{y}} = \quad \hat{\mathbf{z}} \cdot \mathbf{T} \times \hat{\mathbf{z}} = \quad (9h)$$