Homework No. 11 (Fall 2024)

PHYS 500A: MATHEMATICAL METHODS

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Due date: Friday, 2024 Dec 6, 4.30pm

1. (**20 points.**) Inversion is a transformation that maps a point $\mathbf{r}$ inside (outside) a sphere of radius a to a point

$$\mathbf{r}_a = \frac{a^2}{r^2}\mathbf{r} \tag{1}$$

outside (inside) the sphere. Given that the function $\phi(\mathbf{r})$ satisfies the Laplacian,

$$\nabla^2 \phi(\mathbf{r}) = 0, \tag{2}$$

show that

$$\frac{a}{r}\,\phi\left(\frac{a^2}{r^2}\mathbf{r}\right) \tag{3}$$

also satisfies the Laplacian for $r \neq 0$. That is,

$$\nabla^2 \left[\frac{a}{r}\,\phi\left(\frac{a^2}{r^2}\mathbf{r}\right)\right] = 0. \tag{4}$$

To this end, using Eq. (1) evaluate $\mathbf{r}_a \cdot \mathbf{r}_a$ and thus derive

$$r_a r = a^2. \tag{5}$$

Then, show that

$$\frac{a}{r}\,\phi\left(\frac{a^2}{r^2}\mathbf{r}\right) = \frac{r_a}{a}\,\phi(\mathbf{r}_a). \tag{6}$$

To express the gradient in terms of the inverted variable $\mathbf{r}_a$ write

$$\boldsymbol{\nabla} = \frac{\partial}{\partial \mathbf{r}} = \frac{\partial \mathbf{r}_a}{\partial \mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{r}_a} = (\boldsymbol{\nabla}\,\mathbf{r}_a) \cdot \boldsymbol{\nabla}_a. \tag{7}$$

Show that

$$(\boldsymbol{\nabla}\,\mathbf{r}_a) = \frac{1}{a^2}(\mathbf{1}\,r_a^2 - 2\,\mathbf{r}_a\,\mathbf{r}_a). \tag{8}$$

Thus, show that

$$\boldsymbol{\nabla} = \frac{1}{a^2}(\mathbf{1}\,r_a^2 - 2\,\mathbf{r}_a\,\mathbf{r}_a) \cdot \boldsymbol{\nabla}_a \tag{9}$$

and

$$\nabla^2 = \frac{1}{a^4}\left[(\mathbf{1}\,r_a^2 - 2\,\mathbf{r}_a\,\mathbf{r}_a) \cdot \boldsymbol{\nabla}_a\right] \cdot \left[(\mathbf{1}\,r_a^2 - 2\,\mathbf{r}_a\,\mathbf{r}_a) \cdot \boldsymbol{\nabla}_a\right]. \tag{10}$$

Expand the operations and simplify to derive

$$a^4\nabla^2 = r_a^4\nabla_a^2 - 2r_a^2\mathbf{r}_a\cdot\boldsymbol{\nabla}_a. \tag{11}$$

To prove the statement in Eq. (4) show that

$$\nabla^2\left[\frac{a}{r}\,\phi\left(\frac{a^2}{r^2}\mathbf{r}\right)\right] = \frac{r_a^5}{a^5}\nabla_a^2\phi(\mathbf{r}_a) = 0. \tag{12}$$

2. (**20 points.**) The fundamental solution to Laplace's equation is the electric potential due to a point charge,

$$\frac{q}{4\pi\varepsilon_0}\frac{1}{r}. \tag{13}$$

Dropping $q/(4\pi\varepsilon_0)$ we have

$$\nabla^2\frac{1}{r} = 0, \quad r\neq 0. \tag{14}$$

In terms of this solution, we can generate a large number of others. For example, for constant vectors $\mathbf{s}_1$,

$$\nabla^2\left[(\mathbf{s}_1\cdot\boldsymbol{\nabla})\frac{1}{r}\right] = 0, \tag{15}$$

because the gradient operators commute with itself and $\mathbf{s}_1$ is a constant. Solid harmonics of degree $-(l+1)$ are defined as

$$V_l(\mathbf{r}) = \frac{1}{l!}(-\mathbf{s}_1\cdot\boldsymbol{\nabla})(-\mathbf{s}_2\cdot\boldsymbol{\nabla})\ldots(-\mathbf{s}_l\cdot\boldsymbol{\nabla})\frac{1}{r} \tag{16}$$

for $l = 1, 2, \ldots,$ with

$$V_0(\mathbf{r}) = \frac{1}{r} \tag{17}$$

for $l = 0$. Verify that the solid harmonics satisfy the Laplace equation, that is,

$$\nabla^2 V_l(\mathbf{r}) = 0, \qquad l = 0, 1, 2, \ldots. \tag{18}$$

It is insightful to see the explicit form of the solid harmonics after the gradient operations have been evaluated.

(a) Define

$$\mu_i = (\mathbf{s}_i\cdot\hat{\mathbf{r}}), \qquad \tilde{\mu}_i = (\mathbf{s}_i\cdot\mathbf{r}), \tag{19a}$$

$$\lambda_{ij} = (\mathbf{s}_i\cdot\mathbf{s}_j), \qquad \tilde{\lambda}_{ij} = (\mathbf{s}_i\cdot\mathbf{s}_j)r^2. \tag{19b}$$

Show that

$$(-\mathbf{s}_i\cdot\boldsymbol{\nabla})\tilde{\mu}_j = -\frac{\tilde{\lambda}_{ij}}{r^2}, \tag{20a}$$

$$(-\mathbf{s}_i\cdot\boldsymbol{\nabla})\frac{1}{r^m} = \frac{m}{r^{m+2}}\,\tilde{\mu}_i, \tag{20b}$$

$$(-\mathbf{s}_k\cdot\boldsymbol{\nabla})\tilde{\lambda}_{ij} = -\frac{2\tilde{\mu}_k\tilde{\lambda}_{ij}}{r^2}. \tag{20c}$$

(b) Show that

$$V_1 = \frac{1}{1!}\frac{1}{r^3}\Big[\tilde{\mu}_1\Big], \tag{21a}$$

$$V_2 = \frac{1}{2!}\frac{1}{r^5}\Big[3\tilde{\mu}_1\tilde{\mu}_2 - \tilde{\lambda}_{12}\Big], \tag{21b}$$

$$V_3 = \frac{1}{3!}\frac{1}{r^7}\Big[15\tilde{\mu}_1\tilde{\mu}_2\tilde{\mu}_3 - 3\tilde{\mu}_1\tilde{\lambda}_{23} - 3\tilde{\mu}_2\tilde{\lambda}_{31} - 3\tilde{\mu}_3\tilde{\lambda}_{12}\Big], \tag{21c}$$

$$V_4 = \frac{1}{4!}\frac{1}{r^9}\Big[105\tilde{\mu}_1\tilde{\mu}_2\tilde{\mu}_3\tilde{\mu}_4 - 15\tilde{\mu}_1\tilde{\mu}_2\tilde{\lambda}_{34} - 15\tilde{\mu}_1\tilde{\mu}_3\tilde{\lambda}_{24} - 15\tilde{\mu}_1\tilde{\mu}_4\tilde{\lambda}_{23} - 15\tilde{\mu}_2\tilde{\mu}_3\tilde{\lambda}_{14} - 15\tilde{\mu}_2\tilde{\mu}_4\tilde{\lambda}_{13} - 15\tilde{\mu}_3\tilde{\mu}_4\tilde{\lambda}_{12} + 3\tilde{\lambda}_{12}\tilde{\lambda}_{34} + 3\tilde{\lambda}_{13}\tilde{\lambda}_{24} + 3\tilde{\lambda}_{34}\tilde{\lambda}_{12}\Big]. \tag{21d}$$

For bringing compactness we introduce the notation

$$\mu^{l-2m}\lambda^m = \mu_1\mu_2\ldots\mu_{l-2m}\lambda_{..}\lambda_{..}\ldots + \text{combinations} \tag{22}$$

in terms of which we find

$$V_l(\mathbf{r}) = \frac{1}{r^{l+1}}\sum_{m=0}^{l-1}\frac{(2[l-m])!}{2^{l-m}l!(l-m)!}(-1)^m\mu^{l-2m}\lambda^m. \tag{23}$$

(c) Surface (or spherical) harmonics $Y_l(\hat{\mathbf{r}})$ of degree l are defined using the relation

$$V_l(\mathbf{r}) = \frac{Y_l(\hat{\mathbf{r}})}{r^{l+1}}. \tag{24}$$

Show that

$$Y_l(\hat{\mathbf{r}}) = \sum_{m=0}^{l-1}\frac{(2[l-m])!}{2^{l-m}l!(l-m)!}(-1)^m\mu^{l-2m}\lambda^m. \tag{25}$$

(d) Inversion between points $\mathbf{r}$ and $\mathbf{r}_a$ about a sphere of radius a is described by the relations

$$\frac{r_a}{a} = \frac{a}{r} \tag{26}$$

and

$$\mathbf{r}_a = \frac{a^2}{r^2}\mathbf{r} \tag{27}$$

and

$$\mathbf{r} = \frac{a^2}{r_a^2}\mathbf{r}_a. \tag{28}$$

Using inversion we conclude that for every solid harmonic $V_l(\mathbf{r})$ that satisfies the Laplacian there exists another solid harmonic

$$U_l(\mathbf{r}) = \frac{a}{r}V_l\left(\frac{a^2}{r^2}\mathbf{r}\right) \tag{29}$$

that also satisfies the Laplacian. Show that

$$U_1(\mathbf{r}) = \frac{(\mathbf{s}_1 \cdot \mathbf{r})}{a^3}. \tag{30}$$

In general show that

$$U_l(\mathbf{r}) = \frac{r^l}{a^{2l+1}} Y_l(\hat{\mathbf{r}}). \tag{31}$$

Solid harmonics $H_l(\mathbf{r})$ of degree l are defined using the relation

$$H_l(\mathbf{r}) = a^{2l+1} U_l(\mathbf{r}) = r^l Y_l(\hat{\mathbf{r}}). \tag{32}$$

Show that

$$H_l = \sum_{m=0}^{l-1} \frac{(2[l-m])!}{2^{l-m} l!(l-m)!} (-1)^m \tilde{\mu}^{l-2m} \tilde{\lambda}^m. \tag{33}$$

(e) Zonal harmonics $Z_l(\hat{\mathbf{r}})$ of order l are defined to be surface harmonics of degree l with the special choice

$$\mathbf{s}_1 = \mathbf{s}_2 = \cdots = \mathbf{s}_l = \mathbf{s}. \tag{34}$$

Then, $\lambda_{ij} = s^2$ and all μ_i's are identical, say $\mu = (\mathbf{s} \cdot \hat{\mathbf{r}}) = \cos\theta$. Show that

$$Z_l(\hat{\mathbf{r}}) = \sum_{m=0}^{l-1} \frac{(2[l-m])!}{2^l m!(l-m)!(l-2m)!} (-1)^m (\mathbf{s} \cdot \hat{\mathbf{r}})^{l-2m} s^{2m}. \tag{35}$$

Recognize the relation between the zonal harmonics $Z_l(\hat{\mathbf{r}})$ and the Legendre polynomials $P_l(\cos\theta)$ as

$$Z_l(\hat{\mathbf{r}}) = s^l P_l(\cos\theta). \tag{36}$$

3. (**20 points.**) Using the definition of spherical harmonics

$$Y_{lm}(\theta,\phi) = e^{im\phi} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l+m)!}{(l-m)!}} \frac{1}{(\sin\theta)^m} \left(\frac{d}{d\cos\theta}\right)^{l-m} \frac{(\cos^2\theta - 1)^l}{2^l l!}, \tag{37}$$

evaluate the explicit expressions for Y_{00}, Y_{11}, Y_{10}, $Y_{1,-1}$, Y_{22}, Y_{21}, Y_{20}, $Y_{2,-1}$, and $Y_{2,-2}$.

4. (**40 points.**) Generate 3D plots of surface spherical harmonics $Y_{lm}(\theta,\phi)$ as a function of θ and ϕ. In particular,

   (a) Plot $\mathrm{Re}\big[Y_{73}(\theta,\phi)\big]$.

   (b) Plot $\mathrm{Im}\big[Y_{73}(\theta,\phi)\big]$.

   (c) Plot $\mathrm{Abs}\big[Y_{73}(\theta,\phi)\big]$.

   (d) Plot your favourite spherical harmonic, that is, choose a l and m, and Re or Im or Abs.

Hint: In Mathematica these plots are generated using the following commands:

```
SphericalPlot3D[Re[SphericalHarmonicY[l,m,θ,ϕ]],{θ,0,Pi},{ ϕ,0,2 Pi}]
SphericalPlot3D[Im[SphericalHarmonicY[l,m,θ,ϕ]],{θ,0,Pi},{ ϕ,0,2 Pi}]
SphericalPlot3D[Abs[SphericalHarmonicY[l,m,θ,ϕ]],{θ,0,Pi},{ ϕ,0,2 Pi}]
```

Refer diagrams in Wikipedia article on 'spherical harmonics' to see some visual representations of these functions.