Homework No. 01 (2025 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2025 Jan 21, 4.30pm

1. (**20 points.**) Assume Earth to be a solid spherical ball of uniform density. Neglect the influence of all other stars and galaxies. Consider a hypothetical tunnel passing through the center of Earth and connecting two diameterically opposite points on the surface of Earth by a straight line. Ignore friction and the rotational motion of Earth. Use the mass of Earth to be $6.0 \times 10^{24}\,\text{kg}$, radius of Earth to be $6.4 \times 10^{6}\,\text{m}$. Newton's gravitational constant is $6.7 \times 10^{-11}\,\text{Nm}^2/\text{kg}^2$.

   (a) Show that the gravitational field is

$$\mathbf{g}(\mathbf{r}) = \begin{cases} -\hat{\mathbf{r}}\dfrac{GM}{R^2}\dfrac{r}{R}, & \text{for} \quad r < R, \\[2ex] -\hat{\mathbf{r}}\dfrac{GM}{r^2}, & \text{for} \quad R < r, \end{cases} \tag{1}$$

   where G is Newton's gravitational constant and $\hat{\mathbf{r}}$ are unit vectors radiating out from the center of Earth. Plot the magnitude of $\mathbf{g}(\mathbf{r})$ as a function of r. Evaluate the gravitational field on the surface of Earth,

$$g = \frac{GM}{R^2}, \tag{2}$$

   to significant digits, presuming that the gravitational field is continuous at the surface. Note that we could model (undetectable) exotic matter with usual mass to exist only close to the surface of Earth using a δ-function field, which we shall not attempt here.

   (b) The gravitational potential associated with the gravitational field is given by the differential statement

$$\mathbf{g} = -\boldsymbol{\nabla}\phi, \tag{3}$$

   or the integral statement

$$-\,d\phi = \mathbf{g}\cdot d\mathbf{r}. \tag{4}$$

   Thus, determine the gravitational potential

$$\phi(\mathbf{r}) = \begin{cases} \dfrac{1}{2}\dfrac{GM}{R}\dfrac{r^2}{R^2} + c_1, & \text{for} \quad r < R, \\[2ex] -\dfrac{GM}{r} + c_2, & \text{for} \quad R < r. \end{cases} \tag{5}$$

   Determine the arbitrary integration constant c_2 by choosing the gravitational potential to be zero at $r \to \infty$. Requiring the gravitational potential to be continuous at the surface show that

$$c_1 = -\frac{3}{2}\frac{GM}{R}. \tag{6}$$

   Plot the gravitational potential with respect to r using a graphing software. Ponder if there is flexibility here while investigating exotic matter, which we shall not attempt here.

   (c) Consider the free fall from a great distance, say infinity, of an object starting from rest. Determine it's velocity when it reaches the surface of Earth. This is the escape velocity of Earth,

$$v_e = \sqrt{2gR}. \tag{7}$$

(d) Consider a free fall starting from rest at the surface of Earth, in a frictionless tunnel passing through the center of Earth. Find it's velocity as it crosses the center of Earth to be

$$v = \frac{v_e}{\sqrt{2}}. \tag{8}$$

(e) Consider a free fall starting from rest at infinitely large distance falling along a radial line aligned with the tunnel. Find it's velocity as it crosses the center of Earth to be

$$v = v_e \sqrt{\frac{3}{2}}. \tag{9}$$

(f) Is the time taken for these free falls and escape scenarios finite? In particular, try to evaluate the time taken to escape the gravitational field of Earth.

2. (**20 points.**) Motion of a charged particle of mass m and charge q in a uniform magnetic field $\mathbf{B}$ and a uniform electric field $\mathbf{E}$ is governed by

$$m\frac{d\mathbf{v}}{dt} = q\,\mathbf{E} + q\,\mathbf{v} \times \mathbf{B}. \tag{10}$$

Choose $\mathbf{B}$ along the z-axis and $\mathbf{E}$ along the y-axis,

$$\mathbf{B} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + B\,\hat{\mathbf{k}}, \tag{11a}$$
$$\mathbf{E} = 0\,\hat{\mathbf{i}} + E\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}. \tag{11b}$$

Solve this vector differential equation to determine the position $\mathbf{x}(t)$ and velocity $\mathbf{v}(t)$ of the particle as a function of time, for initial conditions

$$\mathbf{x}(0) = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{12a}$$
$$\mathbf{v}(0) = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}. \tag{12b}$$

Verify that the solution is a cycloid characterized by the equations

$$x(t) = R(\omega_c t - \sin \omega_c t), \tag{13a}$$
$$y(t) = R(1 - \cos \omega_c t). \tag{13b}$$

where

$$R = \frac{E}{B\omega_c}, \qquad \omega_c = \frac{qB}{m}. \tag{14}$$

The particle moves as though it were a point on the rim of a wheel of radius R perfectly rolling (without sliding or slipping) with angular speed ω_c along the x-axis. It satisfies the equation of a circle of radius R whose center $(vt, R, 0)$ travels along the x-direction at constant speed v,

$$(x - vt)^2 + (y - R)^2 = R^2, \tag{15}$$

where $v = \omega_c R$.