Homework No. 05 (2025 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2025 Feb 18, 4.30pm

1. (**20 points.**) A two-body system constituting of two masses m_1 and m_2, described by an interaction energy that depends only on the difference in the position of the two bodies

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \tag{1}$$

is described by the Lagrangian

$$L(\mathbf{r}_1, \mathbf{v}_1, \mathbf{r}_2, \mathbf{v}_2) = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - U(\mathbf{r}_1 - \mathbf{r}_2). \tag{2}$$

(a) Derive the canonical momentums,

$$\mathbf{p}_1 = m_1 \mathbf{v}_1 \quad \text{and} \quad \mathbf{p}_2 = m_2 \mathbf{v}_2. \tag{3}$$

Find the Euler-Lagrange equations,

$$\frac{d}{dt}(m_1 \mathbf{v}_1) = -\boldsymbol{\nabla}_1 U(\mathbf{r}_1 - \mathbf{r}_2), \tag{4a}$$

$$\frac{d}{dt}(m_2 \mathbf{v}_2) = \boldsymbol{\nabla}_2 U(\mathbf{r}_1 - \mathbf{r}_2). \tag{4b}$$

Construct the Hamiltonian

$$H(\mathbf{r}_1, \mathbf{p}_1, \mathbf{r}_2, \mathbf{p}_2) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + U(\mathbf{r}_1 - \mathbf{r}_2) \tag{5}$$

and derive the Hamilton equations of motion.

(b) In terms of coordinates representing the position of center of mass

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \tag{6}$$

and the relative position of the masses

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{7}$$

show that

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{m_1 + m_2} \mathbf{r}, \tag{8a}$$

$$\mathbf{r}_2 = \mathbf{R} - \frac{m_1}{m_1 + m_2} \mathbf{r}. \tag{8b}$$

In terms of the respective velocities

$$\mathbf{V} = \frac{d\mathbf{R}}{dt} \quad \text{and} \quad \mathbf{v} = \frac{d\mathbf{r}}{dt} \tag{9}$$

derive

$$\mathbf{v}_1 = \mathbf{V} + \frac{m_2}{m_1 + m_2} \mathbf{v}, \tag{10a}$$

$$\mathbf{v}_2 = \mathbf{V} - \frac{m_1}{m_1 + m_2} \mathbf{v}. \tag{10b}$$

(c) In terms of these coordinates show that

$$L(\mathbf{r}, \mathbf{v}, \mathbf{R}, \mathbf{V}) = \frac{1}{2}MV^2 + \frac{1}{2}\mu v^2 - U(\mathbf{r}), \tag{11}$$

where M is the total mass and μ is the reduced mass,

$$M = m_1 + m_2 \quad \text{and} \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{12}$$

Observe that the reduced mass is always smaller than the smaller of the two masses. Observe that the Lagrangian is independent of $\mathbf{R}$ even though it depends on the associated velocity $\mathbf{V}$. Such a coordinate is called a cyclic coordinate. Determine the canonical momentums,

$$\mathbf{P} = \frac{\partial L}{\partial \mathbf{V}} = M\mathbf{V}, \tag{13a}$$

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \mu\mathbf{v}. \tag{13b}$$

Find the Euler-Lagrange equations of motion,

$$\frac{d}{dt}(M\mathbf{V}) = 0, \tag{14a}$$

$$\frac{d}{dt}(\mu\mathbf{v}) = -\boldsymbol{\nabla}U. \tag{14b}$$

Here the momentum associated with the cyclic coordinate $\mathbf{R}$ is conserved. The canonical momentum associated to a cyclic coordinate is always conserved.

(d) Show that

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 \tag{15}$$

is the toal momentum of the system and

$$\mathbf{p} = \frac{m_2\mathbf{p}_1 - m_1\mathbf{p}_2}{m_1 + m_2} \tag{16}$$

can be defined as the relative momentum. Construct the Hamiltonian using the Legendre transform,

$$H(\mathbf{r}, \mathbf{p}, \mathbf{R}, \mathbf{P}) = \frac{P^2}{2M} + \frac{p^2}{2\mu} + U(\mathbf{r}). \tag{17}$$

Derive the Hamilton equations of motion

$$\frac{d\mathbf{R}}{dt} = \frac{\mathbf{P}}{M}, \tag{18a}$$

$$\frac{d\mathbf{P}}{dt} = 0, \tag{18b}$$

and

$$\frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{\mu}, \tag{19a}$$

$$\frac{d\mathbf{p}}{dt} = -\boldsymbol{\nabla}U. \tag{19b}$$

2. (**20 points.**) The motion of an electric charge of mass m in a time-independent magnetic field, (in the absence of an electric field,) is described by the equation of motion

$$\frac{d}{dt}(m\mathbf{v}) = q\mathbf{v} \times \mathbf{B}. \tag{20}$$

Recall, the electric ad magnetic field is given in terms of (gauge dependent) electric and magnetic potentials as

$$\begin{aligned}\mathbf{E} &= -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad (21)\\ \mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A}. \qquad (22)\end{aligned}$$

The electric potential and magnetic vector potentials are by construction independent of velocity. For time-independent magnetic field, (in the absence of an electric field,) we have

$$\boldsymbol{\nabla}\phi = 0, \qquad \frac{\partial \mathbf{A}}{\partial t} = 0, \qquad (23)$$

and

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \qquad (24)$$

(a) Starting from the equation of motion derive

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = 0. \qquad (25)$$

Thus, kinetic energy is conserved. We shall see that the Hamiltonian for this configuration is numerically equal to the kinetic energy. We emphasize that to satisfy the Hamilton equations of motion the Hamiltonian has to be written in terms of position of momentum, not velocity.

(b) Starting from the equation of motion derive

$$\frac{d}{dt}(m\mathbf{v} + q\mathbf{A}) = \boldsymbol{\nabla}(q\mathbf{v}\cdot\mathbf{A}). \qquad (26)$$

Compare with the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}} = \frac{\partial L}{\partial \mathbf{r}} \qquad (27)$$

and identify

$$\begin{aligned}\frac{\partial L}{\partial \mathbf{v}} &= m\mathbf{v} + q\mathbf{A}, \qquad (28a)\\ \frac{\partial L}{\partial \mathbf{r}} &= \boldsymbol{\nabla}(q\mathbf{v}\cdot\mathbf{A}). \qquad (28b)\end{aligned}$$

Thus, find a Lagrangian

$$L(\mathbf{r},\mathbf{v}) = \frac{1}{2}mv^2 + q\mathbf{v}\cdot\mathbf{A}. \qquad (29)$$

(c) Show that the canonical momentum is given by

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + q\mathbf{A}. \qquad (30)$$

(d) Construct the Hamiltonian using

$$H(\mathbf{r},\mathbf{p}) = \mathbf{p}\cdot\mathbf{v} - L \qquad (31)$$

and show that

$$H(\mathbf{r},\mathbf{p}) = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m}. \qquad (32)$$

Show that the Hamiltonian is equal to kinetic energy, that is,

$$H = \frac{1}{2}mv^2. \qquad (33)$$

(e) Derive the Hamilton equations of motion,

$$\frac{d\mathbf{r}}{dt} = \frac{\partial H}{\partial \mathbf{p}} = \frac{(\mathbf{p} - q\mathbf{A})}{m}, \tag{34}$$

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{r}} = \boldsymbol{\nabla}(q\mathbf{v} \cdot \mathbf{A}). \tag{35}$$

(f) For a constant magnetic field show that

$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r} \tag{36}$$

is a magnetic vector potential upto a gauge. That is, evaluate the curl of $\mathbf{A}$. Show that

$$-q\mathbf{v} \cdot \mathbf{A} = -\frac{q}{2m}\mathbf{B} \cdot \mathbf{L}, \tag{37}$$

where $\mathbf{L} = \mathbf{r} \times (m\mathbf{v})$ is an angular momentum.

3. (**20 points.**) Consider the action,

(a) in terms of the Lagrangian viewpoint,

$$W[\mathbf{x}] = \int_{t_1}^{t_2} dt \left[\frac{1}{2} m \left(\frac{d\mathbf{x}}{dt} \right)^2 - U(\mathbf{x}, t) \right]. \tag{38}$$

Assume no variation at the end points t_1 and t_2. Evaluate the functional derivative

$$\frac{\delta W}{\delta \mathbf{x}(t)}. \tag{39}$$

(b) in terms of the Hamiltonian viewpoint,

$$W[\mathbf{x}, \mathbf{p}] = \int_{t_1}^{t_2} dt \left[\mathbf{p} \cdot \frac{d\mathbf{x}}{dt} - \frac{p^2}{2m} - U(\mathbf{x}, t) \right]. \tag{40}$$

Assume no variation at the end points t_1 and t_2. Evaluate the functional derivatives

$$\frac{\delta W}{\delta \mathbf{x}(t)} \quad \text{and} \quad \frac{\delta W}{\delta \mathbf{p}(t)}. \tag{41}$$

(c) in terms of the Schwingerian viewpoint,

$$W[\mathbf{x}, \mathbf{p}, \mathbf{v}] = \int_{t_1}^{t_2} dt \left[\mathbf{p} \cdot \left(\frac{d\mathbf{x}}{dt} - \mathbf{v} \right) + \frac{1}{2} m v^2 - U(\mathbf{x}, t) \right]. \tag{42}$$

Assume no variation at the end points t_1 and t_2. Evaluate the functional derivatives

$$\frac{\delta W}{\delta \mathbf{x}(t)}, \quad \frac{\delta W}{\delta \mathbf{v}(t)}, \quad \text{and} \quad \frac{\delta W}{\delta \mathbf{p}(t)}. \tag{43}$$