Homework No. 06 (2025 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Thursday, 2025 Feb 27, 4.30pm

1. (**20 points.**) A mass m slides down a frictionless ramp that is inclined at an angle θ with respect to the horizontal. See Fig. 1. Assume uniform gravity g in the vertical downward direction.

   (a) What is the equation of constraint.

   (b) In terms of a suitable dynamical variable write a Lagrangian that describes the motion of the mass.

   (c) Find the equations of motion from the Lagrangian.

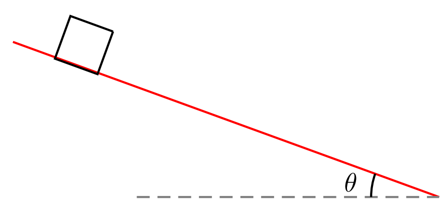

Figure 1: Problem 1.

2. (**20 points.**) The Atwood machine consists of two masses m_1 and m_2 connected by a massless (inextensible) string passing over a massless pulley. See Figure 2. Massless pulley implies that tension in the string on both sides of the pulley is the same, say T. Further, the string being inextensible implies that the magnitude of the accelerations of both the masses are the same. Let $m_2 > m_1$.

   (a) What is the constraint in the variables.

   (b) In terms of a suitable dynamical variable write a Lagrangian that describes the motion of the mass.

   (c) Find the equations of motion from the Lagrangian.

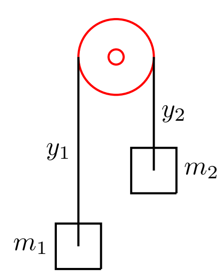

Figure 2: Problem 2.

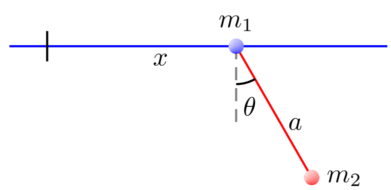

Figure 3: Problem 3.

3. (**20 points.**) A pendulum consists of a mass m_2 hanging from a pivot by a massless string of length a. The pivot, in general, has mass m_1, but, for simplification let $m_1 = 0$. Let the pivot be constrained to move on a horizontal rod. See Figure 3. For simplification, and at loss of generality, let us chose the motion of the pendulum in a vertical plane containing the rod.

   (a) Determine the Lagrangian for the system to be

$$L(x, \dot{x}, \theta, \dot{\theta}) = \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} m_2 a^2 \dot{\theta}^2 + m_2 a \dot{x} \dot{\theta} \cos\theta + m_2 g a \cos\theta. \tag{1}$$

   (b) Evaluate the following derivatives and give physical interpretations of each of these.

$$\frac{\partial L}{\partial \dot{x}} = m_2 \dot{x} + m_2 a \dot{\theta} \cos\theta, \tag{2a}$$

$$\frac{\partial L}{\partial x} = 0, \tag{2b}$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_2 a^2 \dot{\theta} + m_2 a \dot{x} \cos\theta, \tag{2c}$$

$$\frac{\partial L}{\partial \theta} = -m_2 a \dot{x} \dot{\theta} \sin\theta - m_2 g a \sin\theta. \tag{2d}$$

   (c) Determine the equations of motion for the system. Express them in the form

$$\ddot{x} + a\ddot{\theta} \cos\theta - a\dot{\theta}^2 \sin\theta = 0, \tag{3a}$$

$$a\ddot{\theta} + \ddot{x} \cos\theta + g \sin\theta = 0. \tag{3b}$$

   Observe that, like in the case of simple pendulum, the motion is independent of the mass m_2 when $m_1 = 0$.

   (d) In the small angle approximation show that the equations of motion reduce to

$$\ddot{x} + a\ddot{\theta} = 0, \tag{4a}$$

$$a\ddot{\theta} + \ddot{x} + g\theta = 0. \tag{4b}$$

   Determine the solution to be given by

$$\theta = 0 \quad \text{and} \quad \ddot{x} = 0. \tag{5}$$

   Interpret this solution.

   (e) The solution $\theta = 0$ seems to be too restrictive. Will this system not allow $\theta \neq 0$? To investigate this, let us not restrict to the small angle approximation. Rewrite Eqs. (3), using Eq. (3a) in Eq. (3b), as

$$\ddot{x} + a\ddot{\theta} \cos\theta - a\dot{\theta}^2 \sin\theta = 0, \tag{6a}$$

$$\sin\theta \Big[a\ddot{\theta} \sin\theta + a\dot{\theta}^2 \cos\theta + g \Big] = 0. \tag{6b}$$

In this form we immediately observe that $\theta = 0$ is a solution. However, it is not the only solution. Towards interpretting Eqs. (6) let us identify the coordinates of the center of mass of the m_1-m_2 system,

$$(m_1 + m_2)x_{\text{cm}} = m_1 x + m_2(x + a\sin\theta), \tag{7a}$$
$$(m_1 + m_2)y_{\text{cm}} = -m_2 a\cos\theta, \tag{7b}$$

which for $m_1 = 0$ are the coordinates of the mass m_2,

$$x_{\text{cm}} = x + a\sin\theta, \tag{8a}$$
$$y_{\text{cm}} = -a\cos\theta. \tag{8b}$$

Show that

$$\dot{x}_{\text{cm}} = \dot{x} + a\dot{\theta}\cos\theta, \tag{9a}$$
$$\dot{y}_{\text{cm}} = a\dot{\theta}\sin\theta, \tag{9b}$$

and

$$\ddot{x}_{\text{cm}} = \ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta, \tag{10a}$$
$$\ddot{y}_{\text{cm}} = a\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta. \tag{10b}$$

Comparing Eqs. (6) and Eqs. (10) we learn that

$$\ddot{x}_{\text{cm}} = 0, \tag{11a}$$
$$\sin\theta\Big[\ddot{y}_{\text{cm}} + g\Big] = 0. \tag{11b}$$

Thus, $\ddot{y}_{\text{cm}} = -g$ is the more general solution, and $\theta = 0$ is a trivial solution.

(f) Let us analyse the system for initial conditions: $\theta(0) = \theta_0$, $\dot{\theta}(0) = 0$, $\dot{x}(0) = 0$. Show that for this case $\dot{x}_{\text{cm}}(0) = 0$ and

$$a(\cos\theta - \cos\theta_0) = \frac{1}{2}gt^2. \tag{12}$$

Plot θ as a function of time t. Interpret this solution.

(g) **To do:** The interpretation does not seem satisfactory. Is $m_1 = 0$ physical here?

4. (**20 points.**) [Based on Landau and Lifshitz. Section 7.] A particle of mass m moving with velocity $\mathbf{v}_1$ leaves a half-space in which the potential energy is a constant U_1 and enters another in which the potential energy is a different constant $U_2 > U_1$.

(a) Force is the manifestation of the system trying to attain minimum energy. Draw the velocity vector $\mathbf{v}_2$ in Fig. 4 that satisfies these conditions. Does it deflect away from normal or towards the normal?

(b) The potential energy can be described by

$$U(\mathbf{r}) = \begin{cases} U_1, & z < a, \\ U_2, & a < z. \end{cases} \tag{13}$$

In terms of the Heavyside step function

$$\theta(z) = \begin{cases} 0, & z < 0, \\ 1, & 0 < z, \end{cases} \tag{14}$$

show that the potential energy can be expressed in the form

$$U(\mathbf{r}) = U_1 + (U_2 - U_1)\theta(z - a). \tag{15}$$

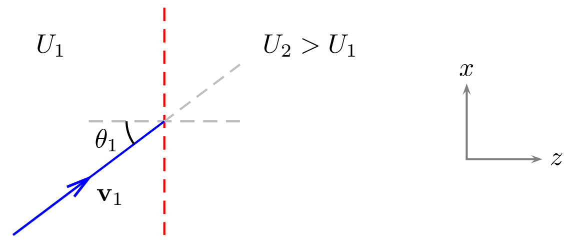

Figure 4: Problem 4.

(c) Show that a suitable Lagrangian for the motion is

$$L(\mathbf{r}, \mathbf{v}) = \frac{1}{2}mv^2 - U_1 - (U_2 - U_1)\theta(z - a). \tag{16}$$

Derive the relations

$$\frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v}, \tag{17a}$$
$$\frac{\partial L}{\partial \mathbf{r}} = -\hat{\mathbf{z}}\,(U_2 - U_1)\,\delta(z - a). \tag{17b}$$

Recall that the derivative of Heaviside step function is a δ-function. Thus, derive the equation of motion

$$\frac{d}{dt}m\mathbf{v} = -\hat{\mathbf{z}}\,(U_2 - U_1)\,\delta(z - a). \tag{18}$$

(d) Show that the momentum in the plane perpendicular to $\hat{\mathbf{z}}$ is conserved. That is,

$$v_1 \sin\theta_1 = v_2 \sin\theta_2. \tag{19}$$

Show that the energy is conserved. That is,

$$\frac{1}{2}mv_1^2 + U_1 = \frac{1}{2}mv_2^2 + U_2. \tag{20}$$

Thus, derive the measure of deflection at the interface to be given by

$$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{1 - \frac{2(U_2 - U_1)}{mv_1^2}}. \tag{21}$$