

Homework No. 08 (2025 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Thursday, 2025 Mar 20, 4.30pm

1. (20 points.) (Refer Landau and Lifshitz, Problem 1 in Chapter 3.) A simple pendulum consists of a particle of mass m suspended by a massless rod of length l in a uniform gravitational field g .

- (a) Identify the two forces acting on the pendulum to be the force of gravity $m\mathbf{g}$ and the force of tension \mathbf{T} . Thus, deduce the Newton equation of motion to be

$$m\mathbf{a} = m\mathbf{g} + \mathbf{T}, \quad (1)$$

where \mathbf{a} is acceleration of mass m . Starting from Eq. (1) derive the equation of motion for the simple pendulum

$$\frac{d^2\phi}{dt^2} = -\omega_0^2 \sin \phi, \quad (2)$$

where

$$\omega_0 = \frac{2\pi}{T_0} = \sqrt{\frac{g}{l}}. \quad (3)$$

- (b) Starting from Eq. (2) derive the statement of conservation of energy for this system,

$$\frac{1}{2}ml^2\dot{\phi}^2 - mgl \cos \phi = \text{constant}. \quad (4)$$

Hint: Multiply Eq. (2) by $\dot{\phi}$ and express the equation as a total derivative with respect to time.

- (c) For initial conditions $\phi(0) = \phi_0$ and $\dot{\phi}(0) = 0$ show that

$$\frac{1}{2}ml^2\dot{\phi}^2 - mgl \cos \phi = -mgl \cos \phi_0. \quad (5)$$

Thus, derive

$$\frac{dt}{T_0} = \frac{1}{2\pi} \frac{d\phi}{\sqrt{2(\cos \phi - \cos \phi_0)}} \quad (6)$$

where $T_0 = 2\pi\sqrt{l/g}$.

- (d) The time period of oscillations of the simple pendulum is equal to four times the time taken between $\phi = 0$ and $\phi = \phi_0$. Thus, show that

$$T = 4 \frac{T_0}{2\pi} \int_0^{\phi_0} \frac{d\phi}{\sqrt{2(\cos \phi - \cos \phi_0)}} \quad (7)$$

$$= \frac{T_0}{\pi} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2 \frac{\phi_0}{2} - \sin^2 \frac{\phi}{2}}}. \quad (8)$$

Then, substitute $\sin \theta = \sin(\phi/2)/\sin(\phi_0/2)$ to determine the period of oscillations of the simple pendulum as a function of the amplitude of oscillations ϕ_0 to be

$$T = T_0 \frac{2}{\pi} K \left(\sin \frac{\phi_0}{2} \right), \quad (9)$$

where

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad (10)$$

is the complete elliptic integral of the first kind.

(e) Using the power series expansion

$$K(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n}(n!)^2} \right]^2 k^{2n} \quad (11)$$

show that for small oscillations ($\phi_0/2 \ll 1$)

$$T = T_0 \left[1 + \frac{\phi_0^2}{16} + \dots \right]. \quad (12)$$

(f) Estimate the percentage error made in the approximation $T \sim T_0$ for $\phi_0 \sim 60^\circ$.

(g) Plot the time period T of Eq. (9) as a function of ϕ_0 . What can you conclude about the time period for $\phi_0 = \pi$?

2. (20 points.) Consider the differential equation

$$\dot{x}(t) = -\omega_1^2 x(t), \quad (13)$$

where dot denotes differentiation with respect to time, in conjunction with a suitable initial condition.

(a) Using Fourier transform

$$x(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{x}(\omega), \quad (14a)$$

$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} x(t), \quad (14b)$$

show that $\tilde{x}(\omega)$ satisfies the algebraic equation

$$-\omega^2 \tilde{x}(\omega) = -\omega_1^2 \tilde{x}(\omega). \quad (15)$$

Observe that we can arrive at this equation using the transcription,

$$\frac{\partial}{\partial t} \rightarrow -i\omega, \quad (16a)$$

$$x(t) \rightarrow \tilde{x}(\omega), \quad (16b)$$

in the original differential equation. Thus, the algebraic equation for $\tilde{x}(\omega)$ is

$$(\omega^2 - \omega_1^2) \tilde{x}(\omega) = 0. \quad (17)$$

(b) The solution to the above algebraic equation can be expressed in the form

$$\tilde{x}(\omega) = \tilde{\alpha}(\omega) \delta(\omega^2 - \omega_1^2), \quad (18)$$

where $\tilde{\alpha}(\omega)$ is to be determined. Using the property of δ -functions show that

$$\tilde{x}(\omega) = \frac{\tilde{\alpha}(\omega_1)}{2\omega_1} \delta(\omega - \omega_1) + \frac{\tilde{\alpha}(-\omega_1)}{2\omega_1} \delta(\omega + \omega_1). \quad (19)$$

(c) Using Fourier transform evaluate

$$x(t) = \frac{1}{2\pi} \frac{\tilde{\alpha}(\omega_1)}{2\omega_1} e^{-i\omega_1 t} + \frac{1}{2\pi} \frac{\tilde{\alpha}(-\omega_1)}{2\omega_1} e^{i\omega_1 t}. \quad (20)$$

In terms of numbers

$$A_1 = \frac{1}{2\pi} \frac{\tilde{\alpha}(\omega_1)}{2\omega_1}, \quad (21a)$$

$$B_1 = \frac{1}{2\pi} \frac{\tilde{\alpha}(-\omega_1)}{2\omega_1}, \quad (21b)$$

express the solution in the form

$$x(t) = A_1 e^{-i\omega_1 t} + B_1 e^{i\omega_1 t}. \quad (22)$$

The numbers A_1 and B_1 are determined from initial conditions. For example, show that for initial conditions $x(0) = A$ and $\dot{x}(0) = 0$ the solution is

$$x(t) = A \cos \omega_1 t. \quad (23)$$

3. (20 points.) Consider the set of differential equations

$$\ddot{x}_1(t) + \omega_1^2 x_1(t) = \omega_3^2 x_2(t), \quad (24a)$$

$$\ddot{x}_2(t) + \omega_2^2 x_2(t) = \omega_3^2 x_1(t), \quad (24b)$$

where dot denotes differentiation with respect to time, in conjunction with suitable initial conditions.

(a) Using Fourier transform show that $\tilde{x}_1(\omega)$ and $\tilde{x}_2(\omega)$ satisfy the algebraic equations

$$(\omega_1^2 - \omega^2)\tilde{x}_1(\omega) - \omega_3^2 \tilde{x}_2(\omega) = 0, \quad (25a)$$

$$-\omega_3^2 \tilde{x}_1(\omega) + (\omega_2^2 - \omega^2)\tilde{x}_2(\omega) = 0. \quad (25b)$$

Observe that they decouple for $\omega_3 = 0$. The explicit nature of the coupling is brought out by writing the solutions, $\tilde{x}_1(\omega)$ and $\tilde{x}_2(\omega)$, in the form

$$\tilde{x}_1(\omega) = \frac{\omega_3^2}{(\omega_1^2 - \omega^2)} \tilde{x}_2(\omega), \quad (26a)$$

$$\tilde{x}_2(\omega) = \frac{\omega_3^2}{(\omega_2^2 - \omega^2)} \tilde{x}_1(\omega). \quad (26b)$$

Using the two solutions in conjunction show that the solutions satisfy

$$(\omega - \lambda_1)(\omega + \lambda_1)(\omega - \lambda_2)(\omega + \lambda_2)\tilde{x}_1(\omega) = 0, \quad (27a)$$

$$(\omega - \lambda_1)(\omega + \lambda_1)(\omega - \lambda_2)(\omega + \lambda_2)\tilde{x}_2(\omega) = 0, \quad (27b)$$

where $\pm\lambda_1$ and $\pm\lambda_2$ are roots of the quartic equation

$$(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) - \omega_3^4 = 0. \quad (28)$$

Evaluate the roots for $\omega_2^2 > \omega_1^2$ to be

$$\lambda_2^2 = \frac{(\omega_2^2 + \omega_1^2)}{2} + \frac{1}{2} \sqrt{(\omega_2^2 - \omega_1^2)^2 + 4\omega_3^4}, \quad (29a)$$

$$\lambda_1^2 = \frac{(\omega_2^2 + \omega_1^2)}{2} - \frac{1}{2} \sqrt{(\omega_2^2 - \omega_1^2)^2 + 4\omega_3^4}, \quad (29b)$$

and express them in the form

$$\lambda_2^2 = \omega_2^2 + (\mu^2 - \Delta^2), \quad (30a)$$

$$\lambda_1^2 = \omega_1^2 - (\mu^2 - \Delta^2), \quad (30b)$$

where

$$\Delta^2 = \frac{(\omega_2^2 - \omega_1^2)}{2} \quad (31)$$

and

$$\mu^2 = \sqrt{\Delta^4 + \omega_3^4}. \quad (32)$$

Determine the normal modes λ_1 and λ_2 for $\omega_3 = 0$.

(b) Derive the following. The difference in the square of roots,

$$\lambda_2^2 - \lambda_1^2 = 2\mu^2, \quad (33)$$

and the change in the normal modes due to coupling,

$$\omega_1^2 - \lambda_1^2 = (\mu^2 - \Delta^2), \quad (34a)$$

$$\omega_1^2 - \lambda_2^2 = -(\mu^2 + \Delta^2), \quad (34b)$$

and

$$\omega_2^2 - \lambda_1^2 = (\mu^2 + \Delta^2), \quad (35a)$$

$$\omega_2^2 - \lambda_2^2 = -(\mu^2 - \Delta^2). \quad (35b)$$

Using the above relations together with

$$\omega_3^2 = \sqrt{(\mu^2 + \Delta^2)(\mu^2 - \Delta^2)} \quad (36)$$

derive

$$\frac{\omega_3^2}{(\omega_1^2 - \lambda_1^2)} = \sqrt{\frac{\mu^2 + \Delta^2}{\mu^2 - \Delta^2}} = -\frac{\omega_3^2}{(\omega_2^2 - \lambda_2^2)}, \quad (37a)$$

$$\frac{\omega_3^2}{(\omega_2^2 - \lambda_1^2)} = \sqrt{\frac{\mu^2 - \Delta^2}{\mu^2 + \Delta^2}} = -\frac{\omega_3^2}{(\omega_1^2 - \lambda_2^2)}. \quad (37b)$$

(c) Argue that the solutions for the algebraic expressions in Eqs. (27) can be expressed in the form

$$\tilde{x}_1(\omega) = \tilde{a}_1(\omega)\delta((\omega - \lambda_1)(\omega + \lambda_1)(\omega - \lambda_2)(\omega + \lambda_2)), \quad (38a)$$

$$\tilde{x}_2(\omega) = \tilde{a}_2(\omega)\delta((\omega - \lambda_1)(\omega + \lambda_1)(\omega - \lambda_2)(\omega + \lambda_2)), \quad (38b)$$

where $\tilde{a}_1(\omega)$ and $\tilde{a}_2(\omega)$ are arbitrary. Using the property of δ -functions show that

$$\frac{\tilde{x}_1(\omega)}{2\pi} = \frac{1}{8\pi\mu^2} \left[\frac{\tilde{a}_1(\lambda_1)}{\lambda_1} \delta(\omega - \lambda_1) + \frac{\tilde{a}_1(-\lambda_1)}{\lambda_1} \delta(\omega + \lambda_1) \right. \\ \left. + \frac{\tilde{a}_1(\lambda_2)}{\lambda_2} \delta(\omega - \lambda_2) + \frac{\tilde{a}_1(-\lambda_2)}{\lambda_2} \delta(\omega + \lambda_2) \right], \quad (39a)$$

$$\frac{\tilde{x}_2(\omega)}{2\pi} = \frac{1}{8\pi\mu^2} \left[\frac{\tilde{a}_2(\lambda_1)}{\lambda_1} \delta(\omega - \lambda_1) + \frac{\tilde{a}_2(-\lambda_1)}{\lambda_1} \delta(\omega + \lambda_1) \right. \\ \left. + \frac{\tilde{a}_2(\lambda_2)}{\lambda_2} \delta(\omega - \lambda_2) + \frac{\tilde{a}_2(-\lambda_2)}{\lambda_2} \delta(\omega + \lambda_2) \right]. \quad (39b)$$

The arbitrary coefficients are related due to the coupling in Eqs. (26). Thus, verify that

$$\tilde{a}_1(\pm\lambda_1) = \sqrt{\frac{\mu^2 + \Delta^2}{\mu^2 - \Delta^2}} \tilde{a}_2(\pm\lambda_1), \quad (40a)$$

$$\tilde{a}_1(\pm\lambda_2) = -\sqrt{\frac{\mu^2 - \Delta^2}{\mu^2 + \Delta^2}} \tilde{a}_2(\pm\lambda_2), \quad (40b)$$

and

$$\tilde{a}_2(\pm\lambda_1) = \sqrt{\frac{\mu^2 - \Delta^2}{\mu^2 + \Delta^2}} \tilde{a}_1(\pm\lambda_1), \quad (41a)$$

$$\tilde{a}_2(\pm\lambda_2) = -\sqrt{\frac{\mu^2 + \Delta^2}{\mu^2 - \Delta^2}} \tilde{a}_1(\pm\lambda_2). \quad (41b)$$

Using Eqs. (39) in the Fourier transform

$$x(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{x}_1(\omega) \quad (42)$$

and the redefinitions

$$A_1 = \frac{1}{8\pi\mu^2} \frac{\tilde{a}_1(\lambda_1)}{\lambda_1}, \quad B_1 = \frac{1}{8\pi\mu^2} \frac{\tilde{a}_1(-\lambda_1)}{\lambda_1}, \quad (43a)$$

$$A_2 = \frac{1}{8\pi\mu^2} \frac{\tilde{a}_1(\lambda_2)}{\lambda_2}, \quad B_2 = \frac{1}{8\pi\mu^2} \frac{\tilde{a}_1(-\lambda_2)}{\lambda_2}, \quad (43b)$$

which are determined by initial conditions, show that

$$x_1(t) = A_1 e^{-i\lambda_1 t} + B_1 e^{i\lambda_1 t} + A_2 e^{-i\lambda_2 t} + B_2 e^{i\lambda_2 t}, \quad (44a)$$

$$x_2(t) = \sqrt{\frac{\mu^2 - \Delta^2}{\mu^2 + \Delta^2}} \left[A_1 e^{-i\lambda_1 t} + B_1 e^{i\lambda_1 t} \right] - \sqrt{\frac{\mu^2 + \Delta^2}{\mu^2 - \Delta^2}} \left[A_2 e^{-i\lambda_2 t} + B_2 e^{i\lambda_2 t} \right]. \quad (44b)$$

(d) For initial conditions

$$x_1(0) = A, \quad \dot{x}_1(0) = 0, \quad x_2(0) = 0, \quad \dot{x}_2(0) = 0, \quad (45)$$

show that

$$x_1(t) = \frac{A}{2} \left[\left(1 + \frac{\Delta^2}{\mu^2} \right) \cos \lambda_1 t + \left(1 - \frac{\Delta^2}{\mu^2} \right) \cos \lambda_2 t \right], \quad (46a)$$

$$x_2(t) = \frac{A}{2} \frac{\omega_3^2}{\mu^2} \left[\cos \lambda_1 t - \cos \lambda_2 t \right]. \quad (46b)$$

Sympathetic oscillations are characterized by the case

$$\Delta^2 \ll \omega_3^2 \quad (47)$$

when

$$\left(1 \pm \frac{\Delta^2}{\mu^2} \right) \sim 1, \quad \frac{\omega_3^2}{\mu^2} \sim 1, \quad \lambda_2^2 \sim \omega_2^2 + \omega_3^2, \quad \lambda_1^2 \sim \omega_1^2 - \omega_3^2, \quad (48)$$

and

$$x_1(t) = \frac{A}{2} \left[\cos \lambda_1 t + \cos \lambda_2 t \right] = A \cos \left(\frac{\lambda_1 - \lambda_2}{2} \right) t \cos \left(\frac{\lambda_1 + \lambda_2}{2} \right) t, \quad (49a)$$

$$x_2(t) = \frac{A}{2} \left[\cos \lambda_1 t - \cos \lambda_2 t \right] = A \sin \left(\frac{\lambda_1 - \lambda_2}{2} \right) t \cos \left(\frac{\lambda_1 + \lambda_2}{2} \right) t. \quad (49b)$$

Plot $x_1(t)$ and $x_2(t)$ for $\omega_2 = 1.01\omega_1$ and $\omega_3 = 0.3\omega_1$, corresponding to $\omega_3 \sim 10\Delta$.

- (e) The following 21 minute long Veritasium YouTube video discusses synchronization in a variety of systems,

https://youtu.be/t-_VPRCtiUg

Discuss if these examples constitute sympathetic oscillations.