Homework No. 10 (2025 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale
Due date: Tuesday, 2025 Apr 8, 4.30pm

1. (**20 points.**) For two functions

$$A = A(\mathbf{x}, \mathbf{p}, t), \tag{1a}$$

$$B = B(\mathbf{x}, \mathbf{p}, t), \tag{1b}$$

the Poisson braket with respect to the canonical variables $\mathbf{x}$ and $\mathbf{p}$ is defined as

$$\big[A, B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} \equiv \frac{\partial A}{\partial \mathbf{x}} \cdot \frac{\partial B}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}} \cdot \frac{\partial B}{\partial \mathbf{x}}. \tag{2}$$

Show that the Poisson braket satisfies the conditions for a Lie algebra. That is, show that

(a) Antisymmetry:

$$\big[A, B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = -\big[B, A\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}. \tag{3}$$

(b) Bilinearity: (a and b are numbers.)

$$\big[aA + bB, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = a\big[A, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + b\big[B, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}. \tag{4}$$

Further show that

$$\big[AB, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = A\big[B, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \big[A, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} B. \tag{5}$$

(c) Jacobi's identity:

$$\Big[A, \big[B, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \Big[B, \big[C, A\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \Big[C, \big[A, B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0. \tag{6}$$

2. (**20 points.**) Show that the commutator of two matrices,

$$\big[\mathbf{A}, \mathbf{B}\big] \equiv \mathbf{AB} - \mathbf{BA}, \tag{7}$$

satisfies the conditions for a Lie algebra, as does the Poisson bracket. In particular show that

(a) Antisymmetry:

$$\big[\mathbf{A}, \mathbf{B}\big] = -\big[\mathbf{B}, \mathbf{A}\big]. \tag{8}$$

(b) Bilinearity: (a and b are numbers.)

$$\big[a\mathbf{A} + b\mathbf{B}, \mathbf{C}\big] = a\big[\mathbf{A}, \mathbf{C}\big] + b\big[\mathbf{B}, \mathbf{C}\big]. \tag{9}$$

Further show that

$$\big[\mathbf{AB}, \mathbf{C}\big] = \mathbf{A}\big[\mathbf{B}, \mathbf{C}\big] + \big[\mathbf{A}, \mathbf{C}\big]\mathbf{B}. \tag{10}$$

(c) Jacobi's identity:

$$\big[\mathbf{A}, \big[\mathbf{B}, \mathbf{C}\big]\big] + \big[\mathbf{B}, \big[\mathbf{C}, \mathbf{A}\big]\big] + \big[\mathbf{C}, \big[\mathbf{A}, \mathbf{B}\big]\big] = 0. \tag{11}$$

3. (**20 points.**) Show that the vector product of two vectors, in this problem denoted using

$$\big[\mathbf{A}, \mathbf{B}\big]_v \equiv \mathbf{A} \times \mathbf{B}, \tag{12}$$

satisfies the conditions for a Lie algebra, as does the Poisson bracket. In particular show that

(a) Antisymmetry:

$$\big[\mathbf{A},\mathbf{B}\big]_v = -\big[\mathbf{B},\mathbf{A}\big]_v. \tag{13}$$

(b) Bilinearity: (a and b are numbers.)

$$\big[a\mathbf{A}+b\mathbf{B},\mathbf{C}\big]_v = a\big[\mathbf{A},\mathbf{C}\big]_v + b\big[\mathbf{B},\mathbf{C}\big]_v. \tag{14}$$

Further show that

$$\big[\mathbf{A}\times\mathbf{B},\mathbf{C}\big]_v = \mathbf{A}\times\big[\mathbf{B},\mathbf{C}\big]_v + \big[\mathbf{A},\mathbf{C}\big]_v\times\mathbf{B}. \tag{15}$$

(c) Jacobi's identity:

$$\big[\mathbf{A},\big[\mathbf{B},\mathbf{C}\big]_v\big]_v + \big[\mathbf{B},\big[\mathbf{C},\mathbf{A}\big]_v\big]_v + \big[\mathbf{C},\big[\mathbf{A},\mathbf{B}\big]_v\big]_v = 0. \tag{16}$$

4. (**20 points.**) (Refer Sec. 21 Dirac's QM book.)
The product rule for Poisson braket can be stated in the following different forms:

$$\big[A_1A_2,B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = A_1\big[A_2,B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \big[A_1,B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}A_2, \tag{17a}$$

$$\big[A,B_1B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = B_1\big[A,B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \big[A,B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}B_2. \tag{17b}$$

(a) Thus, evaluate, in two different ways,

$$\begin{aligned}\big[A_1A_2,B_1B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} &= A_1B_1\big[A_2,B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + A_1\big[A_2,B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}B_2 \\ &\quad +B_1\big[A_1,B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}A_2 + \big[A_1,B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}B_2A_2,\end{aligned} \tag{18a}$$

$$\begin{aligned}\big[A_1A_2,B_1B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} &= B_1A_1\big[A_2,B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + B_1\big[A_1,B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}A_2 \\ &\quad +A_1\big[A_2,B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}B_2 + \big[A_1,B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}A_2B_2.\end{aligned} \tag{18b}$$

(b) Subtracting these results, obtain

$$(A_1B_1 - B_1A_1)\big[A_2,B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \big[A_1,B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}(A_2B_2 - B_2A_2). \tag{19}$$

Thus, using the definition of the commutation relation,

$$\big[A,B\big] \equiv AB - BA, \tag{20}$$

obtain the relation

$$\big[A_1,B_1\big]\big[A_2,B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \big[A_1,B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\big[A_2,B_2\big]. \tag{21}$$

(c) Since this condition holds for A_1 and B_1 independent of A_2 and B_2, conclude that

$$\big[A_1,B_1\big] = i\hbar\big[A_1,B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}, \tag{22a}$$

$$\big[A_2,B_2\big] = i\hbar\big[A_2,B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}, \tag{22b}$$

where $i\hbar$ is necessarily a constant, independent of A_1, A_2, B_1, and B_2. This is the connection between the commutator braket in quantum mechanics and the Poisson braket in classical mechanics. If A's and B's are numbers, then, because their commutation relation is equal to zero, we necessairily have $\hbar = 0$. But, if the commutation relation of A's and B's is not zero, then finite values of $\hbar$ is allowed.

(d) Here the imaginary number $i = \sqrt{-1}$. Show that the constant $\hbar$ is a real number if we presume the Poisson braket to be real, and require the construction

$$C = \frac{1}{i}(AB - BA) \tag{23}$$

to be Hermitian. Experiment dictates that $\hbar = h/2\pi$, where

$$h \sim 6.63\times 10^{-34}\ \text{J}\cdot\text{s} \tag{24}$$

is Planck's constant with dimensions of action.

5. (**20 points.**) Hamiltonian for the motion of a ball (along the radial direction) near the surface of Earth is given by

$$H(z, p_z) = \frac{p_z^2}{2m} - mgz. \tag{25}$$

(a) Determine the equations of motions using

$$\frac{dz}{dt} = \frac{\partial H}{\partial p_z} \qquad \text{and} \qquad \frac{dp_z}{dt} = -\frac{\partial H}{\partial z}. \tag{26}$$

Then, solve the coupled differential equations to find the familiar elementary solution

$$z = z_0 + \frac{p_0}{m}t + \frac{1}{2}gt^2 \tag{27a}$$

and

$$p_z = p_{z,0} + mg\,t. \tag{27b}$$

(b) Next, determine the equations of motion using

$$\frac{dz}{dt} = \big[z, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} \qquad \frac{dp_z}{dt} = \big[p_z, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}. \tag{28}$$

Evaluate

$$\frac{d^2z}{dt^2} = \Big[\big[z, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0}, H\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0} \tag{29a}$$

$$\frac{d^3z}{dt^3} = \Big[\Big[\big[z, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0}, H\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0}, H\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0} \tag{29b}$$

$$\vdots$$

and

$$\frac{d^2p_z}{dt^2} = \Big[\big[p_z, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0}, H\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0} \tag{30a}$$

$$\frac{d^3p_z}{dt^3} = \Big[\Big[\big[p_z, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0}, H\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0}, H\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0} \tag{30b}$$

$$\vdots$$

Then, using

$$z = z_0 + t\left\{\frac{dz}{dt}\right\}_0 + \frac{1}{2}t^2\left\{\frac{d^2z}{dt^2}\right\}_0 + \cdots \tag{31}$$

and

$$p_z = p_{z,0} + t\left\{\frac{dp_z}{dt}\right\}_0 + \frac{1}{2}t^2\left\{\frac{d^2p_z}{dt^2}\right\}_0 + \cdots \tag{32}$$

rederive the solutions in Eqs. (27). Here 0 in the subscripts refers to the initial conditions at $t = 0$.

6. (**20 points.**) Harmonic oscillations are described by the Hamiltonian

$$H(x, p) = \frac{1}{2}p^2 + \frac{1}{2}x^2. \tag{33}$$

(a) Determine the equations of motions using

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} \qquad \text{and} \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x}. \tag{34}$$

Then, solve the coupled differential equations to find the solutions

$$x = x_0 \cos t + p_0 \sin t, \tag{35a}$$
$$p = -x_0 \sin t + p_0 \cos t, \tag{35b}$$

where x_0 and p_0 are given using the intial conditions at $t = 0$.

(b) Next, determine the equations of motion using

$$\frac{dx}{dt} = \big[x, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} \qquad \frac{dp}{dt} = \big[p, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}. \tag{36}$$

Evaluate

$$\Big[\ldots\big[\big[x, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}, \ldots\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} \tag{37}$$

and

$$\Big[\ldots\big[\big[p, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}, \ldots\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} \tag{38}$$

iteratively. Thus, evaluate

$$\frac{d^n x}{dt^n} = \begin{cases} x\, i^n, & \text{for} \quad n = 0, 2, 4, \ldots, \\ p\, i^{n-1}, & \text{for} \quad n = 1, 3, 5, \ldots, \end{cases} \tag{39}$$

and

$$\frac{d^n p}{dt^n} = \begin{cases} p\, i^n, & \text{for} \quad n = 0, 2, 4, \ldots, \\ -x\, i^{n-1}, & \text{for} \quad n = 1, 3, 5, \ldots. \end{cases} \tag{40}$$

Then, using

$$x = x_0 + t\left\{\frac{dx}{dt}\right\}_0 + \frac{1}{2}t^2\left\{\frac{d^2x}{dt^2}\right\}_0 + \cdots \tag{41}$$

and

$$p = p_0 + t\left\{\frac{dp}{dt}\right\}_0 + \frac{1}{2}t^2\left\{\frac{d^2p}{dt^2}\right\}_0 + \cdots \tag{42}$$

rederive the solutions in Eqs. (35). Here 0 in the subscripts refers to the initial conditions at $t = 0$.

7. (**20 points.**) Given F and G are constants of motion, that is

$$\big[F, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0 \qquad \text{and} \qquad \big[G, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0. \tag{43}$$

Then, using Jacobi's identity, show that $\big[F, G\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}$ is also a constant of motion. Thus, conclude the following:

(a) If L_x and L_y are constants of motion, then L_z is also a constant of motion.

(b) If p_x and L_z are constants of motion, then p_y is also a constant of motion.