

Homework No. 11 (2025 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2025 Apr 15, 4.30pm

1. (20 points.) (Refer Schwinger's QM, chapter 9) The Hamiltonian for a Kepler problem is

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{\alpha}{|\mathbf{r}_1 - \mathbf{r}_2|}, \quad (1)$$

where \mathbf{r}_1 and \mathbf{r}_2 are the positions of the two constituent particles of masses m_1 and m_2 .

- (a) Introduce the coordinates representing the center of mass, relative position, total momentum, and relative momentum:

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \quad \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \quad \mathbf{p} = \frac{m_2\mathbf{p}_1 - m_1\mathbf{p}_2}{m_1 + m_2}, \quad (2)$$

respectively, to rewrite the Hamiltonian as

$$H = \frac{P^2}{2M} + \frac{p^2}{2\mu} - \frac{\alpha}{r}, \quad (3)$$

where

$$M = m_1 + m_2, \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \quad (4)$$

- (b) Show that Hamilton's equations of motion are given by

$$\frac{d\mathbf{R}}{dt} = \frac{\mathbf{P}}{M}, \quad \frac{d\mathbf{P}}{dt} = 0, \quad \frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{\mu}, \quad \frac{d\mathbf{p}}{dt} = -\frac{\alpha\mathbf{r}}{r^3}. \quad (5)$$

- (c) Verify that the Hamiltonian H , the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and the Laplace-Runge-Lenz vector

$$\mathbf{A} = \frac{\mathbf{r}}{r} - \frac{\mathbf{p} \times \mathbf{L}}{\mu\alpha}, \quad (6)$$

are the three constants of motion for the Kepler problem. That is, show that

$$\frac{dH}{dt} = 0, \quad \frac{d\mathbf{L}}{dt} = 0, \quad \frac{d\mathbf{A}}{dt} = 0. \quad (7)$$

2. (20 points.) In the Kepler problem the orbit of a planet is a conic section

$$r(\phi) = \frac{r_0}{1 + e \cos(\phi - \phi_0)} \quad (8)$$

expressed in terms of the eccentricity e and distance r_0 . Determine the constant ϕ_0 to be 0 by requiring the initial condition

$$r(0) = \frac{r_0}{1 + e}. \quad (9)$$

This leads to

$$r(\pi) = \frac{r_0}{1 - e}. \quad (10)$$

The distance r_0 is characterized by the fact that the effective potential

$$U_{\text{eff}}(r) = \frac{L_z^2}{2\mu r^2} - \frac{\alpha}{r} \quad (11)$$

is minimum at r_0 . We used the definitions

$$r_0 = \frac{L_z^2}{\mu\alpha}, \quad U_{\text{eff}}(r_0) = -\frac{\alpha}{2r_0}, \quad e = \sqrt{1 - \frac{E}{U_{\text{eff}}(r_0)}}. \quad (12)$$

Thus, the orbit of a planet is completely determined by the energy E and the angular momentum L_z , which are constants of motion. The statement of conservation of angular momentum can be expressed in the form

$$dt = \frac{\mu}{L_z} r^2 d\phi, \quad (13)$$

which is convenient for evaluating the time elapsed in the motion. For the case of elliptic orbit, $U_{\text{eff}}(r_0) < E < 0$, show that the time period is given by

$$T = \frac{\mu}{L_z} \int_0^{2\pi} d\phi \frac{r_0^2}{(1 + e \cos \phi)^2} = \frac{\mu r_0^2}{L_z} \frac{2\pi}{(1 - e^2)^{\frac{3}{2}}}. \quad (14)$$

Show that at point ‘2’ in Figure 2

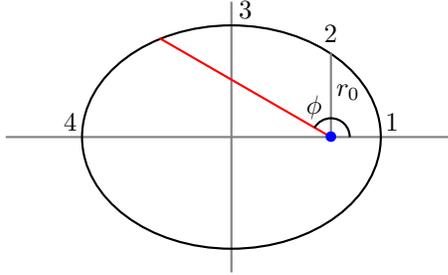


Figure 1: Elliptic orbit

$$\phi = \frac{\pi}{2}, \quad \text{and} \quad r = r_0. \quad (15)$$

The time taken to go from ‘1’ to ‘2’ is given by (need not be proved here)

$$t_{1 \rightarrow 2} = \frac{\mu}{L_z} \int_0^{\frac{\pi}{2}} d\phi \frac{r_0^2}{(1 + e \cos \phi)^2} = \frac{T}{4} \left(\frac{4}{\pi} \tan^{-1} \sqrt{\frac{1-e}{1+e}} - \frac{2e}{\pi} \sqrt{1-e^2} \right). \quad (16)$$

Evaluate $t_{1 \rightarrow 2}$ for $e = 0$ and $e = 1$. Show that at point ‘3’ in Figure 2

$$\phi = \pi - \tan^{-1} \left(\frac{\sqrt{1-e^2}}{e} \right), \quad \text{and} \quad r = a. \quad (17)$$

The time taken to go from ‘1’ to ‘3’ is given by (need not be proved here)

$$t_{1 \rightarrow 3} = \frac{\mu}{L_z} \int_0^{\pi - \tan^{-1} \left(\frac{\sqrt{1-e^2}}{e} \right)} d\phi \frac{r_0^2}{(1 + e \cos \phi)^2} = \frac{T}{4} \left(1 - \frac{2e}{\pi} \right). \quad (18)$$

Similarly, the time taken to go from '3' to '4' is given by (need not be proved here)

$$t_{3 \rightarrow 4} = \frac{\mu}{L_z} \int_{\pi - \tan^{-1}\left(\frac{\sqrt{1-e^2}}{e}\right)}^{\pi} d\phi \frac{r_0^2}{(1 + e \cos \phi)^2} = \frac{T}{4} \left(1 + \frac{2e}{\pi}\right). \quad (19)$$

Evaluate the time elapsed in the above cases for $e \rightarrow 0$ and $e \rightarrow 1$. The eccentricity e of Earth's orbit is 0.0167 and timeperiod T is 365 days. Thus, calculate

$$t_{1 \rightarrow 3} - t_{1 \rightarrow 2} \quad (20)$$

for Earth in units of days.