Homework No. 12 (2025 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Thursday, 2025 May 8, 4.30pm

1. (**20 points.**) (Resource: Lecture from [2024S].)

   Kepler problem is described by the potential energy

$$U(r) = -\frac{\alpha}{r}, \tag{1}$$

and the corresponding Lagrangian

$$L(\mathbf{r}, \mathbf{v}) = \frac{1}{2}\mu v^2 + \frac{\alpha}{r}. \tag{2}$$

For the case when the total energy E is negative,

$$-\frac{\alpha}{2r_0} < E < 0, \qquad r_0 = \frac{L_z^2}{\mu\alpha}, \tag{3}$$

where L_z is the angular momentum, the motion is described by an ellipse,

$$r(\phi) = \frac{r_0}{1 + e\cos(\phi - \phi_0)}, \qquad e = \sqrt{1 + \frac{E}{(\alpha/2r_0)}}. \tag{4}$$

Perihelion is the point in the orbit of a planet when it is closest to the Sun. This corresponds to $\phi = \phi_0$. The precession of the perihelion is suitably defined in terms of the angular displacement $\Delta\phi$ of the perihelion during one revolution,

$$\Delta\phi = 2\left[\int_{r_{\min}}^{r_{\max}} d\phi\right] - 2\pi, \tag{5}$$

where one revolution is defined as twice the transition between points when the planet is closest and farthest from Sun in terms of

$$r_{\min} = \frac{r_0}{1 + e} \tag{6}$$

the perihelion, when the planet is closest to Sun, and

$$r_{\max} = \frac{r_0}{1 - e} \tag{7}$$

is the aphelion, corresponding to $\phi = \phi_0 + \pi$, when the planet is farthest from Sun.

   (a) For the Kepler problem derive the relation

$$d\phi = \frac{r_0\, dr}{r^2} \frac{1}{\sqrt{e^2 - \left(1 - \frac{r_0}{r}\right)^2}}. \tag{8}$$

   Show that the precession of perihelion is zero for the Kepler problem.

   (b) When a small correction

$$\delta U(r) = -\frac{\beta}{r^3} = \kappa\, U_0 \left(\frac{r_0}{r}\right)^3, \tag{9}$$

expressed in terms of dimensionless parameter κ using the relation $\beta = -\kappa U_0 r_0^3$, is added we have the perturbed potential energy

$$U(r) = -\frac{\alpha}{r} - \frac{\beta}{r^3} = -\frac{\alpha}{2r_0}\left[\frac{r_0}{r} + \kappa\left(\frac{r_0}{r}\right)^3\right]. \tag{10}$$

Show that the precession of the perihelion due to this perturbation is

$$\Delta\phi = -3\pi\kappa = -\frac{6\pi\beta}{\alpha r_0^2}. \tag{11}$$