Homework No. 13 (2025 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale
Due date: Thursday, 2025 Apr 24, 4.30pm

1. (**20 points.**) The relativity principle states that the laws of physics are invariant (or covariant) when observed using different coordinate systems. In special relativity we restrict these coordinate systems to be uniformly moving with respect to each other. Let $z = z' = 0$ at $t = 0$.

   (a) Linear: Spatial homogeneity, spatial isotropy, and temporal homogeneity, require the transformation to be linear. (We will skip this derivation.) Then, for simplicity, restricting to coordinate systems moving with respect to each other in a single direction, we can write

$$z' = A(v)\, z + B(v)\, t, \tag{1a}$$
$$t' = E(v)\, z + F(v)\, t. \tag{1b}$$

   We will refer to the respective frames as primed and unprimed.

   (b) Identity: An object P at rest in the primed frame, described by $z' = 0$, will be described in the unprimed frame as $z = vt$.

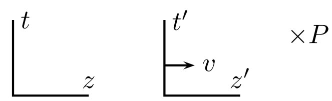

Figure 1: Identity.

   Using these in Eq. (1a), we have

$$0 = A(v)\, vt + B(v)\, t. \tag{2}$$

   This implies $B(v) = -vA(v)$. Thus, show that

$$z' = A(v)\,(z - vt), \tag{3a}$$
$$t' = E(v)\, z + F(v)\, t. \tag{3b}$$

   (c) Reversal: The descriptions of a process in the unprimed frame moving to the right with velocity v with respect to the primed should be identical to those made in the unprimed (with their axis flipped) moving with velocity $-v$ with respect to the primed (with their axis flipped). This is equivalent to the requirement of isotropy in an one dimensional space.

Figure 2: Reversal.

   That is, the transformation must be invariant under

$$z \to -z, \qquad z' \to -z', \qquad v \to -v. \tag{4}$$

This implies

$$-z' = A(-v)\,(-z+vt), \tag{5a}$$
$$t' = -E(-v)\,z + F(-v)\,t. \tag{5b}$$

Show that Eqs. (3a) and (5a) in conjunction imply

$$A(-v) = A(v). \tag{6}$$

Further, show that Eqs. (3b) and (5b) in conjunction implies

$$E(-v) = -E(v), \tag{7a}$$
$$F(-v) = F(v). \tag{7b}$$

(d) Reciprocity: The description of a process in the unprimed frame moving to the right with velocity v is identical to the description in the primed frame moving to the left.

Figure 3: Reciprocity.

That is, the transformation must be invariant under

$$(z,t) \to (z',t') \qquad (z',t') \to (z,t) \qquad v \to -v. \tag{8}$$

Show that this implies

$$z = A(-v)\,(z'+vt'), \tag{9a}$$
$$t = E(-v)\,z' + F(-v)\,t'. \tag{9b}$$

Show that Eqs. (3) and Eqs. (9) imply

$$E(v) = \frac{1}{v}\left[\frac{1}{A(v)} - A(v)\right], \tag{10a}$$
$$F(v) = A(v). \tag{10b}$$

(e) Together, for arbitrary $A(v)$, show that the relativity principle allows the following transformations,

$$z' = A(v)\,(z-vt), \tag{11a}$$
$$t' = A(v)\left[\frac{1}{v}\left(\frac{1}{A(v)^2} - 1\right)z + t\right]. \tag{11b}$$

i. In Galilean relativity we require $t' = t$. Show that this is obtained with

$$A(v) = 1 \tag{12}$$

in Eqs. (11). This leads to the Galilean transformation

$$z' = z - vt, \tag{13a}$$
$$t' = t. \tag{13b}$$

ii. In Einstein's special relativity the requirement is for a special speed c that is described identically by both the primed and unprimed frames. That is,

$$z = ct, \tag{14a}$$
$$z' = ct'. \tag{14b}$$

Show that Eqs. (14) when substituted in in Eqs. (11) leads to

$$A(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{15}$$

This corresponds to the Lorentz transformation

$$z' = A(v)(z - vt), \tag{16a}$$
$$t' = A(v)\left(-\frac{v}{c^2}z + t\right). \tag{16b}$$

iii. This suggests that it should be possible to contrive additional solutions for $A(v)$ that respects the relativity principle, but with new physical requirements for the respective choice of $A(v)$. Construct one such transformation. In particular, investigate modifications of Eqs. (14) that donot change the current experimental observations. The response to this part of the question will not be used for assessment.

2. (**20 points.**) Lorentz transformation describing a boost in the x-direction, y-direction, and z-direction, are

$$L_1 = \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 & 0 & 0 \\ -\beta_1\gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} \gamma_2 & 0 & -\beta_2\gamma_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\beta_2\gamma_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} \gamma_3 & 0 & 0 & -\beta_3\gamma_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_3\gamma_3 & 0 & 0 & \gamma_3 \end{pmatrix}, \tag{17}$$

respectively. Transformation describing a rotation about the x-axis, y-axis, and z-axis, are

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\omega_1 & \sin\omega_1 \\ 0 & 0 & -\sin\omega_1 & \cos\omega_1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega_2 & 0 & -\sin\omega_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\omega_2 & 0 & \cos\omega_2 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega_3 & \sin\omega_3 & 0 \\ 0 & -\sin\omega_3 & \cos\omega_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{18}$$

respectively. For infinitesimal transformations, $\beta_i = \delta\beta_i$ and $\omega_i = \delta\omega_i$ use the approximations

$$\gamma_i \sim 1, \qquad \cos\omega_i \sim 1, \qquad \sin\omega_i \sim \delta\omega_i, \tag{19}$$

to identify the generator for boosts $\mathbf{N}$, and the generator for rotations the angular momentum $\mathbf{J}$,

$$\mathbf{L} = \mathbf{1} + \delta\boldsymbol{\beta}\cdot\mathbf{N} \qquad \text{and} \qquad \mathbf{R} = \mathbf{1} + \delta\boldsymbol{\omega}\cdot\mathbf{J}, \tag{20}$$

respectively. Then derive

$$\Big[N_1, N_2\Big] = N_1N_2 - N_2N_1 = J_3. \tag{21}$$

This states that boosts in perpendicular direction leads to rotation. (To gain insight of the statement, calculate $[J_1, J_2]$ and interpret the result.)

(a) Is velocity addition commutative?

(b) Is velocity addition associative?

(c) Read a resource article (Wikipedia) on Wigner rotation.

3. (**20 points.**) (Based on Hughston and Tod's book.) Prove the following.

   (a) If p^μ is a time-like vector and $p^\mu s_\mu = 0$ then s^μ is necessarily space-like.

   (b) If p^μ and q^μ are both time-like vectors and $p^\mu q_\mu < 0$ then either both are future-pointing or both are past-pointing.

   (c) If p^μ and q^μ are both light-like vectors and $p^\mu q_\mu = 0$ then p^μ and q^μ are proportional.

   (d) If p^μ is a light-like vector and $p^\mu s_\mu = 0$, then s^μ is space-like or p^μ and s^μ are proportional.

   (e) If u^α, v^α, and w^α, are time-like vectors with $u^\alpha v_\alpha < 0$ and $v^\alpha w_\alpha < 0$, then $w^\alpha u_\alpha < 0$.

4. (**20 points.**) The Poincaré formula for the addition of (parallel) velocities is

$$v = \frac{v_a + v_b}{1 + \dfrac{v_a v_b}{c^2}}, \tag{22}$$

   where v_a and v_b are velocities and c is speed of light in vacuum. Jerzy Kocik, from the department of Mathematics in SIUC, has invented a geometric diagram that allows one to visualize the Poincaré formula. (Refer [1].) An interactive applet for exploring velocity addition is available at Kocik's web page [2]. (For the following assume that the Poincaré formula holds for all speeds, subluminal ($v_i < c$), superluminal ($v_i > c$), and speed of light.)

   (a) Analyse what is obtained if you add two subluminal speeds?

   (b) Analyse what is obtained if you add a subluminal speed to speed of light?

   (c) Analyse what is obtained if you add a subluminal speed to a superluminal speed?

   (d) Analyse what is obtained if you add speed of light to another speed of light?

   (e) Analyse what is obtained if you add a superluminal speed to speed of light?

   (f) Analyse what is obtained if you add two superluminal speeds?

References

[1] J. Kocik. Geometric diagram for relativistic addition of velocities. *Am. J. Phys.*, 80:737–739, August 2012.

[2] J. Kocik. An interactive applet for exploring relativistic velocity addition. Link to Webpage.