Midterm Exam No. 02 (Fall 2025)

PHYS 500A: MATHEMATICAL METHODS

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1. (**20 points.**) The Pauli matrix

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

is written in the eigenbasis of

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

Write σ_x in the eigenbasis of itself.

2. (**20 points.**) Consider the inhomogeneous linear differential equation

$$\left(\frac{d^2}{dt^2} + 2\gamma \frac{d}{dt} + \omega_0^2 \right) x(t) = \delta(t). \tag{3}$$

Use the Fourier transformation and the associated inverse Fourier transformation

$$x(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{x}(\omega), \tag{4a}$$

$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} x(t), \tag{4b}$$

to find $\tilde{x}(\omega)$.

3. (**20 points.**) A mass m moving in a straight line is described by Newton's second law,

$$m \frac{d^2 x}{dt^2} = F(t), \tag{5}$$

where $x(t)$ is the position of mass and $F(t)$ is force. Let $v(t) = dx/dt$. If initial conditions are set out at a moment infinitesimally before $t = 0$,

$$x(0-) = 0, \tag{6}$$

$$v(0-) = v_0, \tag{7}$$

determine the motion for a force given by

$$F(t) = -m v_0\, \delta(t). \tag{8}$$

4. (**20 points.**) Evaluate the integral

$$\int_{-1}^{1} dx \frac{(5x^3 - 3x)}{\sqrt{1 + t^2 - 2tx}}. \tag{9}$$

Recall

$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x. \tag{10}$$

5. (**20 points.**) Spherical harmonics or surface harmonics of degree l are

$$Y_l(\mathbf{r}) = \sqrt{\frac{2l+1}{4\pi}} \frac{r^{l+1}}{l!} (-\mathbf{s}_1 \cdot \boldsymbol{\nabla})(-\mathbf{s}_2 \cdot \boldsymbol{\nabla}) \ldots (-\mathbf{s}_l \cdot \boldsymbol{\nabla}) \frac{1}{r} \tag{11}$$

for $l = 0, 1, 2, \ldots,$ with

$$Y_0(\mathbf{r}) = \sqrt{\frac{1}{4\pi}}, \tag{12}$$

where $\mathbf{s}_l$ are constant vectors. Here constant refers to independent of position $\mathbf{r}$. Recall that the fundamental solution to Laplace's equation is the electric potential due to a point charge,

$$\frac{q}{4\pi\varepsilon_0} \frac{1}{r}. \tag{13}$$

Dropping $q/(4\pi\varepsilon_0)$ we have

$$\nabla^2 \frac{1}{r} = 0, \quad r \neq 0, \tag{14}$$

where $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. Zonal harmonics $P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{z}})$ of degree l are defined in terms of spherical harmonics of degree l for the choice

$$\mathbf{s}_1 = \mathbf{s}_2 = \cdots = \mathbf{s}_l = \hat{\mathbf{z}}, \tag{15}$$

as

$$P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{z}}) = \sqrt{\frac{4\pi}{2l+1}} Y_l(\hat{\mathbf{r}}). \tag{16}$$

In terms of

$$\mu = \hat{\mathbf{z}} \cdot \hat{\mathbf{r}} = \hat{\mathbf{z}} \cdot \boldsymbol{\nabla} r = \frac{\partial r}{\partial z} = \frac{z}{r} \tag{17}$$

show that

$$P_l(\mu) = \frac{r^{l+1}}{l!} \left(-\frac{\partial}{\partial z} \right)^l \frac{1}{r}. \tag{18}$$

Evaluate the zonal harmonics of degree $l = 0, 1, 2, 3$.