Homework No. 10 (Fall 2025)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Monday, 2025 Nov 17, 4.30pm

0. Problems 2, 4, and 6, are for submission.

1. (**20 points.**) The generating function for the spherical harmonics, $Y_{lm}(\theta, \phi)$, is

$$\frac{1}{l!}\left(\mathbf{a}\cdot\frac{\mathbf{r}}{r}\right)^l = \sum_{m=-l}^{l}\sqrt{\frac{4\pi}{2l+1}}\,Y_{lm}(\theta,\phi)\,\psi_{lm}, \tag{1}$$

where the left hand side is expressed in terms of position vector $\mathbf{r}$ and a null vector $\mathbf{a}$,

$$\mathbf{r} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{2}$$

$$\mathbf{a} = \frac{1}{2}(y_-^2 - y_+^2, -iy_-^2 - iy_+^2, 2y_-y_+), \tag{3}$$

and the right hand side consists of

$$\psi_{lm} = \frac{y_+^{l+m}}{\sqrt{(l+m)!}}\frac{y_-^{l-m}}{\sqrt{(l-m)!}} \tag{4}$$

and

$$Y_{lm}(\theta,\phi) = e^{im\phi}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!}{(l-m)!}}\frac{1}{(\sin\theta)^m}\left(\frac{d}{d\cos\theta}\right)^{l-m}\frac{(\cos^2\theta-1)^l}{2^l l!}. \tag{5}$$

An example of a null-vector is

$$\mathbf{a} = (-i\cos\alpha, -i\sin\alpha, 1). \tag{6}$$

(a) Identify the corresponding $y_\pm$ in Eq. (3) to show that, now, ψ_{lm} in Eq. (1) is

$$\psi_{lm} = \frac{e^{-im\left(\alpha-\frac{\pi}{2}\right)}}{\sqrt{(l+m)!(l-m)!}}. \tag{7}$$

(b) Then, integrate Eq. (1) to derive an integral representation for spherical harmonics,

$$\frac{1}{l!}\int_0^{2\pi}\frac{d\alpha}{2\pi}\,e^{im\alpha}\big[\cos\theta - i\sin\theta\cos(\phi-\alpha)\big]^l = \sqrt{\frac{4\pi}{2l+1}}\frac{i^m\,Y_{lm}(\theta,\phi)}{\sqrt{(l+m)!(l-m)!}}. \tag{8}$$

(c) By setting $m = 0$ derive the corresponding integral representation for Legendre polynomial $P_l(\cos\theta)$:

$$\int_0^\pi \frac{d\alpha}{\pi}\left[\cos\theta - i\sin\theta\cos\alpha\right]^l = P_l(\cos\theta). \tag{9}$$

2. (**20 points.**) Given

$$\left(\frac{a}{r} + \frac{\partial}{\partial r}\right)\left(\frac{b}{r} + \frac{\partial}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}. \tag{10}$$

Find the numbers a and b.

3. (**20 points.**) [Differential equation for spherical harmonics.]
Polynomials $(\mathbf{a}\cdot\mathbf{r})^l$ of degree l satisfy the Laplacian when $\mathbf{a}$ is a null-vector, that is,

$$(\mathbf{a}\cdot\mathbf{a}) = 0. \tag{11}$$

(a) Show that

$$\nabla^2(\mathbf{a}\cdot\mathbf{r})^l = l(l-1)(\mathbf{a}\cdot\mathbf{r})^{(l-2)}(\mathbf{a}\cdot\mathbf{a}), \tag{12}$$

and conclude

$$\nabla^2(\mathbf{a}\cdot\mathbf{r})^l = 0. \tag{13}$$

(b) Write the polynomial construction in the form

$$(\mathbf{a}\cdot\mathbf{r})^l = r^l(\mathbf{a}\cdot\hat{\mathbf{r}})^l. \tag{14}$$

Observe that $(\mathbf{a}\cdot\hat{\mathbf{r}})^l$ has no radial dependence. Thus, in this form, the radial and angular dependence is separated. Starting from the Laplacian in spherical polar coordinates,

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right](\mathbf{a}\cdot\mathbf{r})^l = 0, \tag{15}$$

deduce

$$\frac{r^l}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right](\mathbf{a}\cdot\hat{\mathbf{r}})^l + (\mathbf{a}\cdot\hat{\mathbf{r}})^l\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}r^l = 0. \tag{16}$$

(c) Show that

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}r^l = l(l+1)\frac{r^l}{r^2}. \tag{17}$$

Thus, derive the differential equation for the generating function

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + l(l+1)\right](\mathbf{a}\cdot\hat{\mathbf{r}})^l = 0. \tag{18}$$

(d) Use the generating function

$$\frac{(\mathbf{a}\cdot\hat{\mathbf{r}})^l}{l!} = \sum_{m=-l}^{l} \psi_{lm} \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta,\phi) \tag{19}$$

written in terms of

$$\psi_{lm} = \frac{y_+^{l+m}}{\sqrt{(l+m)!}} \frac{y_-^{l-m}}{\sqrt{(l-m)!}} \tag{20}$$

to derive the differential equation for spherical harmonics

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + l(l+1)\right] Y_{lm}(\theta,\phi) = 0. \tag{21}$$

4. (**20 points.**) Spherical harmonics satisfy the differential equations

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + l(l+1)\right] Y_{lm}(\theta,\phi) = 0. \tag{22}$$

Verify this explicitly for $l = 0, 1$, and all possible values of m.

5. (**20 points.**) [Orthogonality conditions for spherical harmonics.]
For a null-vector $\mathbf{a}$, that satisfies

$$\mathbf{a}\cdot\mathbf{a} = 0, \tag{23}$$

the polynomial $(\mathbf{a}\cdot\hat{\mathbf{r}})^l$ of degree l is the generating function of spherical harmonics $Y_{lm}(\theta,\phi)$. To derive the orthonormality properties of spherical harmonics let us consider the product of two generating functions, with null-vectors $\mathbf{a}$ and $\mathbf{a}^*$, integrated over all the angles,

$$\int d\Omega\, (\mathbf{a}^*\cdot\hat{\mathbf{r}})^l (\mathbf{a}\cdot\hat{\mathbf{r}})^{l'}, \tag{24}$$

where

$$d\Omega = \sin\theta d\theta d\phi. \tag{25}$$

(a) After integration over the angles the product of the two generating functions is a scalar. Thus, it has to be constructed out of $(\mathbf{a}\cdot\mathbf{a})$, $(\mathbf{a}^*\cdot\mathbf{a}^*)$, and $(\mathbf{a}^*\cdot\mathbf{a})$. Since $(\mathbf{a}\cdot\mathbf{a}) = 0$ and $(\mathbf{a}^*\cdot\mathbf{a}^*) = 0$, the integral has to be constructed out of $(\mathbf{a}^*\cdot\mathbf{a})$. This is possible only if $l = l'$. Together, we conclude

$$\int d\Omega\, (\mathbf{a}^*\cdot\hat{\mathbf{r}})^l (\mathbf{a}\cdot\hat{\mathbf{r}})^{l'} = \delta_{ll'} (\mathbf{a}^*\cdot\mathbf{a})^l C_l, \tag{26}$$

in terms of arbitrary constant C_l.

(b) To determine C_l choose

$$\mathbf{a} = (1, i, 0). \tag{27}$$

For this choice of null-vector, evaluate $\mathbf{a}^* = (1, -i, 0)$, $(\mathbf{a} \cdot \hat{\mathbf{r}}) = \sin\theta e^{i\phi}$, $(\mathbf{a}^* \cdot \hat{\mathbf{r}}) = \sin\theta e^{-i\phi}$, and $(\mathbf{a}^* \cdot \hat{\mathbf{a}}) = 2$. Thus, find

$$C_l = \frac{4\pi}{2^l} \int_0^1 dt (1 - t^2)^l, \tag{28}$$

after substituting $\cos\theta = t$. Evaluate

$$C_0 = 4\pi. \tag{29}$$

Integrate by parts in the integral for C_l to derive the recurrence relation

$$C_l = \frac{l}{2l+1} C_{l-1}. \tag{30}$$

Evaluate

$$C_l = \frac{4\pi 2^l l! l!}{(2l+1)!}. \tag{31}$$

Thus, conclude

$$\int d\Omega \, \frac{(\mathbf{a}^* \cdot \hat{\mathbf{r}})^l}{l!} \frac{(\mathbf{a} \cdot \hat{\mathbf{r}})^{l'}}{l!} = \delta_{ll'} 4\pi \frac{(\mathbf{a}^* \cdot \mathbf{a})^l 2^l}{(2l+1)!}. \tag{32}$$

(c) For null-vectors constructed out of $y_\pm$ in the form

$$\mathbf{a} = \left(\frac{y_-^2 - y_+^2}{2}, \frac{y_-^2 + y_+^2}{2i}, y_+ y_- \right) \tag{33}$$

show that

$$4\pi \frac{(\mathbf{a}^* \cdot \mathbf{a})^l 2^l}{(2l+1)!} = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} \sum_{m'=-l'}^{l'} \psi_{lm}^* \psi_{l'm'} \delta_{mm'}, \tag{34}$$

where

$$\psi_{lm} = \frac{y_+^{l+m}}{\sqrt{(l+m)!}} \frac{y_-^{l-m}}{\sqrt{(l-m)!}}. \tag{35}$$

Using the generating function

$$\frac{(\mathbf{a}^* \cdot \hat{\mathbf{r}})^l}{l!} = \sum_{m=-l}^{l} \psi_{lm} \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta, \phi) \tag{36}$$

show that

$$\sum_{m=-l}^{l} \sum_{m'=-l'}^{l'} \psi_{lm}^* \psi_{l'm'} \sqrt{\frac{4\pi}{2l+1}} \sqrt{\frac{4\pi}{2l'+1}} \int d\Omega \, Y_{lm}^*(\theta, \phi) Y_{l'm'}(\theta, \phi)$$

$$= \delta_{ll'} \sqrt{\frac{4\pi}{2l+1}} \sqrt{\frac{4\pi}{2l'+1}} \sum_{m=-l}^{l} \sum_{m'=-l'}^{l'} \psi_{lm}^* \psi_{l'm'} \delta_{mm'}. \tag{37}$$

Thus, comparing the two sides of the equality, read out the orthonormality condition for the spherical harmonics,

$$\int d\Omega\, Y^*_{lm}(\theta,\phi) Y_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'}. \tag{38}$$

6. (**20 points.**) Spherical harmonics satisfy the orthogonality conditions

$$\int d\Omega\, Y^*_{lm}(\theta,\phi) Y_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'} \tag{39}$$

Verify this explicitly for $l = 0, 1$, and all possible values of m.