Homework No. 11 (Fall 2025)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Friday, 2025 Dec 5, 4.30pm

1. (**20 points.**) Use the integral representation of $J_m(t)$,

$$i^m J_m(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{it\cos\alpha - im\alpha}, \tag{1}$$

to prove the recurrence relations

$$2\frac{d}{dt}J_m(t) = J_{m-1}(t) - J_{m+1}(t), \tag{2a}$$

$$2\frac{m}{t}J_m(t) = J_{m-1}(t) + J_{m+1}(t). \tag{2b}$$

2. (**10 points.**) Using the recurrence relations of Eq. (2), show that

$$\left(-\frac{d}{dt} + \frac{m-1}{t}\right)\left(\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = \left(\frac{d}{dt} + \frac{m+1}{t}\right)\left(-\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = J_m(t) \tag{3}$$

and from this derive the differential equation satisfied by $J_m(t)$.

3. (**20 points.**) Using the recurrence relations,

$$2\frac{d}{dt}J_m(t) = J_{m-1}(t) - J_{m+1}(t), \tag{4a}$$

$$2\frac{m}{t}J_m(t) = J_{m-1}(t) + J_{m+1}(t), \tag{4b}$$

satisfied by the Bessel functions, derive the 'ladder' operations satisfied by the Bessel functions,

$$\left(\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = J_{m-1}(t), \tag{5}$$

$$\left(-\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = J_{m+1}(t). \tag{6}$$

In quantum mechanics a ladder operator is a raising or lowering operator that transforms eigenfunctions by increasing or decreasing the eigenvalue.

4. (**10 points.**) Integral representations for the modified Bessel functions, $I_m(t)$ and $K_m(t)$, for integer m and $0 \le t < \infty$ are

$$K_m(t) = \int_0^\infty d\theta\, \cosh m\theta\, e^{-t\cosh\theta}, \tag{7a}$$

$$I_m(t) = \int_0^\pi \frac{d\phi}{\pi}\, \cos m\phi\, e^{t\cos\phi}. \tag{7b}$$

(a) Using Mathematica (or your favourite graphing tool) plot $K_0(t), K_1(t), K_2(t)$ and $I_0(t), I_1(t), I_2(t)$ on the same plot. (Please do not submit hand sketched plots.)

(b) Refer Chapter 10 of Digital Library of Mathematical Functions,

https://dlmf.nist.gov/10

for a comprehensive resource.

5. (**10 points.**) Show that the integral representations for the modified Bessel functions, $I_m(t)$ and $K_m(t)$, for integer m and $0 \le t < \infty$,

$$K_m(t) = \int_0^\infty d\theta\, \cosh m\theta\, e^{-t\cosh\theta}, \tag{8a}$$

$$I_m(t) = \int_0^\pi \frac{d\phi}{\pi}\, \cos m\phi\, e^{t\cos\phi}. \tag{8b}$$

satisfies the differential equation for modified Bessel functions,

$$\left[-\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + \frac{m^2}{t^2} + 1\right]\left\{\begin{matrix} I_m(t) \\ K_m(t) \end{matrix}\right\} = 0. \tag{9}$$

Hint: Integrate by parts, after identifying

$$\left(t\cosh\theta - t^2\sinh^2\theta\right) e^{-t\cosh\theta} = -\frac{d^2}{d\theta^2}e^{-t\cosh\theta}, \tag{10a}$$

$$\left(t\cos\phi - t^2\sin^2\phi\right) e^{t\cos\phi} = -\frac{d^2}{d\phi^2}e^{t\cos\phi}. \tag{10b}$$

6. (**20 points.**) The cylindrical free Green's function satisfies

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{m^2}{\rho^2} + k_z^2\right] g_m(\rho,\rho';k_z) = \frac{\delta(\rho-\rho')}{\rho}. \tag{11}$$

Integrate Eq. (11) around $\rho = \rho'$ to derive the continuity conditions:

$$g_m(\rho,\rho';k_z)\Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = 0, \tag{12a}$$

$$\rho\frac{\partial}{\partial\rho} g_m(\rho,\rho';k_z)\Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = -1. \tag{12b}$$

Let us further require that

$$g_m(0, \rho'; k_z) \text{ is finite,} \tag{13a}$$

$$g_m(\infty, \rho'; k_z) = 0. \tag{13b}$$

Recall the Wronskian

$$I_m(t)K'_m(t) - I'_m(t)K_m(t) = -\frac{1}{t}. \tag{14}$$

Construct the solution to have the form

$$g_m(\rho, \rho') = \begin{cases} A\, I_m(k_z\rho) + B\, K_m(k_z\rho), & 0 \leq \rho < \rho', \\ C\, I_m(k_z\rho) + D\, K_m(k_z\rho), & \rho' < \rho < \infty. \end{cases} \tag{15}$$

Derive the solution

$$g_m(\rho, \rho') = I_m(k_z\rho_<)K_m(k_z\rho_>), \tag{16}$$

where $\rho_< = \text{Minimum}(\rho, \rho')$ and $\rho_> = \text{Maximum}(\rho, \rho')$.