

## Homework No. 13 (2026 Spring)

### PHYS 510: CLASSICAL MECHANICS

*School of Physics and Applied Physics, Southern Illinois University–Carbondale*

Due date: Tuesday, 2026 Apr 28, 4.30pm

1. (20 points.) Lorentz transformation describing a boost in the  $x$ -direction,  $y$ -direction, and  $z$ -direction, are

$$L_1 = \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 & 0 & 0 \\ -\beta_1\gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} \gamma_2 & 0 & -\beta_2\gamma_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\beta_2\gamma_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} \gamma_3 & 0 & 0 & -\beta_3\gamma_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_3\gamma_3 & 0 & 0 & \gamma_3 \end{pmatrix}, \quad (1)$$

respectively. Transformation describing a rotation about the  $x$ -axis,  $y$ -axis, and  $z$ -axis, are

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\omega_1 & \sin\omega_1 \\ 0 & 0 & -\sin\omega_1 & \cos\omega_1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega_2 & 0 & -\sin\omega_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\omega_2 & 0 & \cos\omega_2 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega_3 & \sin\omega_3 & 0 \\ 0 & -\sin\omega_3 & \cos\omega_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2)$$

respectively. For infinitesimal transformations,  $\beta_i = \delta\beta_i$  and  $\omega_i = \delta\omega_i$  use the approximations

$$\gamma_i \sim 1, \quad \cos\omega_i \sim 1, \quad \sin\omega_i \sim \delta\omega_i, \quad (3)$$

to identify the generator for boosts  $\mathbf{N}$ , and the generator for rotations the angular momentum  $\mathbf{J}$ ,

$$\mathbf{L} = \mathbf{1} + \delta\boldsymbol{\beta} \cdot \mathbf{N} \quad \text{and} \quad \mathbf{R} = \mathbf{1} + \delta\boldsymbol{\omega} \cdot \mathbf{J}, \quad (4)$$

respectively. Then derive

$$[N_1, N_2] = N_1N_2 - N_2N_1 = J_3. \quad (5)$$

This states that boosts in perpendicular direction leads to rotation. (To gain insight of the statement, calculate  $[J_1, J_2]$  and interpret the result.)

- (a) Is velocity addition commutative?
  - (b) Is velocity addition associative?
  - (c) Read a resource article (Wikipedia) on Wigner rotation.
2. (20 points.) (Based on Hughston and Tod's book.) Prove the following.
- (a) If  $p^\mu$  is a time-like vector and  $p^\mu s_\mu = 0$  then  $s^\mu$  is necessarily space-like.
  - (b) If  $p^\mu$  and  $q^\mu$  are both time-like vectors and  $p^\mu q_\mu < 0$  then either both are future-pointing or both are past-pointing.
  - (c) If  $p^\mu$  and  $q^\mu$  are both light-like vectors and  $p^\mu q_\mu = 0$  then  $p^\mu$  and  $q^\mu$  are proportional.
  - (d) If  $p^\mu$  is a light-like vector and  $p^\mu s_\mu = 0$ , then  $s^\mu$  is space-like or  $p^\mu$  and  $s^\mu$  are proportional.
  - (e) If  $u^\alpha$ ,  $v^\alpha$ , and  $w^\alpha$ , are time-like vectors with  $u^\alpha v_\alpha < 0$  and  $v^\alpha w_\alpha < 0$ , then  $w^\alpha u_\alpha < 0$ .

3. (20 points.) The Poincaré formula for the addition of (parallel) velocities is

$$v = \frac{v_a + v_b}{1 + \frac{v_a v_b}{c^2}}, \quad (6)$$

where  $v_a$  and  $v_b$  are velocities and  $c$  is speed of light in vacuum. Jerzy Kocik, from the department of Mathematics in SIUC, has invented a geometric diagram that allows one to visualize the Poincaré formula. (Refer [1].) An interactive applet for exploring velocity addition is available at Kocik's web page [2]. (For the following assume that the Poincaré formula holds for all speeds, subluminal ( $v_i < c$ ), superluminal ( $v_i > c$ ), and speed of light.)

- (a) Analyse what is obtained if you add two subluminal speeds?
- (b) Analyse what is obtained if you add a subluminal speed to speed of light?
- (c) Analyse what is obtained if you add a subluminal speed to a superluminal speed?
- (d) Analyse what is obtained if you add speed of light to another speed of light?
- (e) Analyse what is obtained if you add a superluminal speed to speed of light?
- (f) Analyse what is obtained if you add two superluminal speeds?

## References

- [1] J. Kocik. Geometric diagram for relativistic addition of velocities. *Am. J. Phys.*, 80:737–739, August 2012.
- [2] J. Kocik. An interactive applet for exploring relativistic velocity addition. [Link to Webpage](#).